Surfaces with $\kappa = 0$

If S minimal surface with $\kappa(S)=0$, then $K_S^2 = 0$

$\chi(\mathcal{O}_S) \geq 0$

$e(S) \geq 0$

Moreover, for all $n \geq 1$, we have

$$P_n(S) \in \{0, 1\}$$

$$\& \quad \exists\, n_0 \geq 1 \text{ s.t. } P_{n_0}(S) = 1.$$

$$\chi(\mathcal{O}_S) = 1 - q + p_g \geq 0 \Rightarrow 1 \geq p_g \geq q - 1$$

$$\Rightarrow q \leq 2.$$

So we have only the following possibilities:

(1) $q = 0$, $p_g = 1$ (K3 surfaces)

(2) $q = 0$, $p_g = 0$ (Enriques surfaces)

(3) $q = 2$, $p_g = 1$ (abelian surfaces)

(4) $q = 1$, $p_g = 0$ (bielliptic surfaces)

(5) $q = 1$, $p_g = 1$ (cannot occur)

Case 1) $q = 0$, $p_g = 1$

Thm If S minimal with $\kappa = 0$, $q = 0$, $p_g = 1$ then K_S is trivial, hence all pluricanonical system are trivial & base point free

Pf.

$$h^0(2K_S) + h^0(-K_S) \geq \chi(2K_S) = \chi(\mathcal{O}_S) + K_S^2 = \underset{\substack{\| \\ 2}}{\chi(\mathcal{O}_S)}$$

$$\left.\begin{array}{l} P_2(S) = h^0(2K_S) \leq 1 \Rightarrow h^0(-K_S) \geq 1 \\ p_g = h^0(K_S) = 1 \end{array}\right\} \Rightarrow K_S \text{ trivial.}$$

Examples

adjunction formula : $\overset{\text{eff div.}}{Y} \subset \overset{\text{sm}}{X} \Rightarrow \omega_Y \simeq (\omega_X \otimes \mathcal{O}_X(Y))|_Y$

- Kodaira vanishing $\Rightarrow$ If $n \geq 3$, Y smooth & $\mathcal{O}_X(Y)$ ample then $h^1(X, \mathcal{O}_X) = h^1(Y, \mathcal{O}_Y)$
- $Y = Y_1 \cap \cdots \cap Y_k$ complete intersection $\Rightarrow \omega_Y \simeq (\omega_X \otimes \mathcal{O}_X(Y_1 + \cdots + Y_k))|_Y$
- If Y smooth of $\dim \geq 2$, $Y_1, \cdots, Y_k$ ample then $h^1(X, \mathcal{O}_X) = h^1(Y, \mathcal{O}_Y)$

$S \subset \mathbb{P}^n$ Complete intersection of hypersurfaces $H_1, \cdots, H_{n-2}$ (deg $d_1, \cdots, d_{n-2}$, resp)

$\Rightarrow \omega_S \cong \mathcal{O}_S(\Sigma d_i - n - 1)$ & $h^1(S, \mathcal{O}_S) = 0$

K3 surfaces: examples

Smooth quartic surface $X_4 \subset \mathbb{P}^3$

Smooth complete intersection of type $(2,3)$

$$X_2 \cap X_3 \subset \mathbb{P}^4$$

Smooth complete intersection of type $(2,2,2)$

$$X_2 \cap X_2' \cap X_2'' \subset \mathbb{P}^5$$

double plane branched along a smooth sextic curve

$$\begin{array}{ccl} X & \xrightarrow[\pi]{2:1} & \mathbb{P}^2 \\ & & \cup \\ & & C \text{ sextic} \end{array}$$

$\mathcal{O}_{\mathbb{P}^2}(C) \cong \delta^{\otimes 2} \in \mathrm{Pic}(\mathbb{P}^2) \cong \mathbb{Z}$

$K_X = \pi^*(K_{\mathbb{P}^2} \otimes \delta)$

Case 2) $q = 0$, $p_g = 0$.

Thm S minimal with $\chi(S) = 0$, $q = 0$, $p_g = 0$

then $2K_S$ is trivial

hence all even pluricanonical systems are trivial & bpf

all odd pluricanonical systems $\cong$ the same nontrivial order 2 line bundle.

& hence $P_{2n+1}(S) = 0$, $\forall n \geq 0$

Pf Step 1: $P_2 = 1$.

If $P_2 = 0 \xRightarrow{\text{Castelnuovo rationality}} S$ rational $\not\leftarrow$.

$$h^0(-2K_S) + h^0(3K_S) \geq \chi(-2K_S) \overset{R\text{-}R.}{=} \underset{\substack{\| \\ 1}}{\chi(\mathcal{O}_S)}$$

Step 2: $P_3 = 0$

If $P_3 = 1$. then $\exists!$ curve $D \in |3K_S|$

$\exists!$ curve $D' \in |2K_S|$

Say $D = \sum_{i=1}^{k} a_i D_i$ $\quad D' = \sum_{i=1}^{k} b_i D_i$ $\quad \begin{cases} D_i : \text{distinct irred. curves} \\ a_i, b_i \geq 0 \end{cases}$

$\Rightarrow P_6 = 1$, unique curve in $|6K_S|$ is $2D = 3D'$

$$\Rightarrow \quad 2\sum_{i=1}^{k} a_i D_i = 3\sum_{i=1}^{k} b_i D_i$$

$$\Rightarrow \quad 2a_i = 3b_i, \quad \forall\, 1 \le i \le k$$

$\exists$ non-negative integers $\lambda_1, \dots, \lambda_k$ s.t.

$$a_i = 3\lambda_i \quad \& \quad b_i = 2\lambda_i \quad \forall\, i.$$

$$\Rightarrow \quad D - D' = \sum_{i=1}^{k}(a_i - b_i)D_i = \sum_{i=1}^{k} \lambda_i D_i \in |K_S|$$

i.e. $p_g = 1$ $\lightning$.

Step 3 (End)

$$\left.\begin{array}{l} P_3 = 0 \Longrightarrow h^0(-2K_S) \geq 1 \\ P_2(S) = h^0(2K_S) = 1 \end{array}\right\} \Rightarrow 2K_S \text{ trivial.}$$

$\square$

Case 3) $q = 2$, $p_g = 1$.

We need the following special case of Poincaré complete reducibility theorem

<u>lemma</u> $\begin{array}{c} A \\ \cup \\ C \end{array}$ A abelian surface, C smooth elliptic curve

then $\exists$ smooth elliptic curve E & a morph. $f : A \longrightarrow E$ w/ connected fibres such that C is a fibre of f.

<u>Pf.</u> Up to translations, <u>WMA</u> C contains the origin $0 \in A$

K_A trivial $\Rightarrow C^2 = C(C + K_A) = 0$

For $\forall\, x \in A$, consider the translated curve $C_x := x + C$, $C_x \cong C$

- $0 \in C \Rightarrow x \in C_x$
- Clearly, $C_x \underset{\text{hom.eq.}}{\sim} C \Rightarrow C_x \cdot C = C^2 = 0$, $C_x \cdot C = C_x^2$

then given $x, y \in A$, either $C_x = C_y$

or $C_x \cap C_y = \emptyset$

Moreover, for $\forall z \in A$,

C_z is the unique curve in the family $\mathcal{C} := \{C_x\}_{x \in A}$

passing through z

- if $y \in C_x$, then $y \in C_x \cap C_y$
$$\Downarrow$$
$$C_x = C_y$$

Now consider the map

$$f : A \longrightarrow \operatorname{Pic}^0(A)$$
$$x \longmapsto [C_x - C]$$

then all curves in the family $\mathcal{C}$ are contained in fibres of f

<u>Claim</u>: f non-constant

(Otherwise, all curves in $\mathcal{C}$ were linearly equiv. $\rightsquigarrow$ a morph. $\phi : A \to \mathbb{P}^1$ s.t. curves in $\mathcal{C}$ are fibres of φ $\xRightarrow{\text{CBF}}$ $K_A = \phi^* \mathcal{O}_{\mathbb{P}^1}(-2)$ $\lightning$)

$\Rightarrow$ the image of f is a curve E,

up to Stein factorization,

$$\begin{tikzcd} & B \arrow[dr, "\text{finite}"] & \\ f : A \arrow[ur, "\text{fib.}"] \arrow[rr] & & E \subset \operatorname{Pic}^0(A) \end{tikzcd}$$

<u>WMA</u> E smooth & $\mathcal{C}$ is the family of all fibres of f

$$g(E) < \underset{\substack{\| \\ 2}}{q(A)} \leq g(E) + 1 \Rightarrow g(E) = 1$$

$\square$

<u>Thm (Enriques theorem)</u>

S minimal surf with $\chi(S) = 0$, $q = 2$, $P_g = 1$.

then S is an abelian surface

<u>Pf</u>. Consider the Albanese morph.

$$a : S \longrightarrow A := \operatorname{Alb} S$$

(Case 1) $\dim a(S) = 1$ put $a(S) := C$

then $a : S \longrightarrow C$ is a fibration over a smooth curve C of genus $q = 2$ (fibration of genus g)

$0 = \chi(\mathcal{O}_S) \geq (g-1)(g(C)-1) = g-1 \Rightarrow g \leq 1$

If $g = 0$, then S ruled $\Rightarrow \chi(S) = -\infty$ $\lightning$

$\Rightarrow g = 1$.

In the case, the fibration ∂ is smooth & isotrivial

then $\exists$ étale cover $B \xrightarrow{\pi} C$ s.t.

$$\begin{tikzcd}
S' := B\times F \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & S \arrow[r, dashed, "\overset{bir}{\simeq}"] \arrow[d] & (B\times F)/G \arrow[d] \\
B \arrow[r, "\pi"'] & C \arrow[r, "\sim"] & B/G
\end{tikzcd}$$

$$\underset{\substack{\| \\ 2}}{q(S)} = \underset{\substack{\| \\ g(C)=2}}{g(B/G)} + g(F/G) \Rightarrow g(F/G)=0$$

Now take a non-trivial torsion 2 point $\delta \in Pic^0(C)$

& Consider the étale double cover $\pi : \tilde{C} \to C$

$$2g(\tilde{C})-2 = 2(2g(C)-2) \Rightarrow g(\tilde{C})=3$$

Consider the Cartesian diagram

$$\begin{tikzcd}
\tilde{S} \arrow[r, "\tilde{\pi}"] \arrow[d, "\tilde{\partial}"'] \arrow[dr, phantom, "\square"] & S \arrow[d, "\partial"] \\
\tilde{C} \arrow[r, "\pi"'] & C
\end{tikzcd}$$

<u>claim</u>: that $\kappa(\tilde{S}) = \kappa(S)$.

- $K_{\tilde{S}} = \tilde{\pi}^* K_S \Rightarrow \kappa(\tilde{S}) \geq \kappa(S)$.

- take $n \gg 0$ s.t.

$$\dim \operatorname{im}(\varphi_{|nK_{\tilde{S}}|}) = \kappa(\tilde{S}).$$

$S = \tilde{S}/\mathbb{Z}_2$ free $\mathbb{Z}_2$-quotient.

$$\Rightarrow H^0(S, nK_S) = H^0(\tilde{S}, nK_{\tilde{S}})^{\mathbb{Z}_2} \qquad G := \mathbb{Z}_2$$

take any non-zero section $t \in H^0(\tilde{S}, nK_{\tilde{S}})$ &

Consider the section $t^G \in H^0(\tilde{S}, 2nK_{\tilde{S}})$ defined as

$$t^G(x) := \prod_{h\in G} t(h(x)) \quad \text{for } \forall\, x \in \tilde{S}.$$

$$\Rightarrow t^G \neq 0 \ \& \ t^G \in H^0(\tilde{S}, 2nK_{\tilde{S}})^G \simeq H^0(S, 2nK_S)$$

Now consider the map

$$\begin{aligned}
\rho : \mathbb{P}(H^0(\tilde{S}, nK_{\tilde{S}})) &\longrightarrow \mathbb{P}(H^0(\tilde{S}, 2nK_{\tilde{S}})^G) = \mathbb{P}(H^0(S, 2nK_S)) \\
[(s)] &\longmapsto [(s^G)]
\end{aligned}$$

the map ρ is finite since $\rho : \underset{\text{general}}{D} \in |nK_{\tilde{S}}| \mapsto D^G := \sum_{\gamma\in G} \gamma(D) \in |2nK_{\tilde{S}}|$

$$\Rightarrow \dim(\operatorname{Im} \varphi_{2nK_S}) = \dim(\operatorname{Im} \varphi_{nK_{\tilde{S}}}) = \kappa(\tilde{S}) \quad \square$$

Hence $\kappa(\tilde{S})=0 \Rightarrow q(\tilde{S})\le 2$

but $q(\tilde{S})\ge g(\tilde{C})=3$ $\lightning$

(Case 2) $a: S \twoheadrightarrow A$

Step 1: K_S trivial.

Otherwise, if K_S not trivial, let $D=\sum_{i=1}^{k} a_i D_i \in |K_S|$ unique element

$$0=K_S^2=K_S D \ge K_S D_i = D D_i = \sum_{j\ne i} a_j D_j D_i + a_i D_i^2 \ge a_i D_i^2$$

$$\Rightarrow p_a(D_i)=1+\frac{1}{2}(K_S+D_i)D_i \le 1$$

$\Rightarrow$ either D_i smooth elliptic curve

or D_i is rational curve, either smooth or singular with a node or a cusp.

Assume D_i rational for all $1\le i\le k$.

then each divisor D_i contracted to a point by a $\quad S \xrightarrow{a} A$ abelian surf, $D_i \subset S$

$\Rightarrow$ D contracted to a union of points.

$\Rightarrow K_S^2=D^2<0$ $\lightning$

$\Rightarrow \exists i$ s.t. D_i is a smooth elliptic curve.

$$\left.\begin{cases} D_i^2\le 0 \\ K_S D_i \le 0 \\ (K_S+D_i)D_i=0 \end{cases}\right\} \Rightarrow D_i^2=0=K_S D_i$$

$\Downarrow$

D_i not contracted to a point by a

$\Downarrow$

its image $E:=a(D_i)$ is a smooth elliptic curve in A

$\Downarrow$ Lemma

$\exists$ elliptic curve B & morphism $f: A\to B$ with connected fibres s.t. E is a fibre

$E \subset A$, $E \longmapsto pt$

$$\beta: S \xrightarrow{a} A \xrightarrow{f} B$$
$$D_i \longmapsto E \longmapsto pt$$

$\Rightarrow D_i \subseteq$ a fibre of β, $D_i^2=0$ $\Big\} \Rightarrow D_i$ is the support of a fibre of β

If mD_i is the full fibre of β, then for $\forall l\in\mathbb{Z}_{>0}$

$$h^0(S, lnK_S) \ge h^0(S, lnD_i)=l \;\lightning$$

$\square$

Case 4) $q=1$, $p_g=0$

Thm If S surf with $\kappa(S)=0$, $q=1$, $p_g=0$

& for $n\in\mathbb{Z}_{>0}$, $P_n(S)=1$

then nK_S is trivial.

Pf. Consider the Albanese morph.

$$a: S \longrightarrow \underset{\text{elliptic curve}}{A := \operatorname{Alb} S}$$

it is a fibration with general fibre F

$\Rightarrow g(F)\geq 1$.

Claim: $g(F)=1$.

Indeed, assume $g(F)\geq 2$.

$$\deg a_*\omega_{S/A} = \chi(\mathcal{O}_S) - (g(A)-1)(g(F)-1) = 0$$

$\Rightarrow \exists$ étale cover $C \xrightarrow{\pi} A$ s.t. we have a Cartesian square

$$\begin{tikzcd} S' \arrow[r, "\pi'"] \arrow[d, "a'"'] \arrow[dr, phantom, "\lrcorner"] & S \arrow[d, "a"] \\ C \arrow[r, "\pi"] & A \end{tikzcd}$$

with C elliptic curve & $S' = C\times F$, $g(F)\geq 2$ $\Big\} \Rightarrow \kappa(S')=1$.

by the same argument as Enriques theorem's proof.

$$\kappa(S) = \kappa(S') = 1 \quad ↯.$$

Step 2: let $n\in\mathbb{Z}_{>0}$ s.t. $P_n=1$

Consider the unique div. $D\in|nK_S|$

claim: $D=0$.

assume $D\neq 0$.

- F elliptic $\Rightarrow K_SF=0 \Rightarrow DF=0 \Rightarrow D$ union of fibres of a
- $D^2=(nK_S)^2=0 \overset{\text{Zariski lem.}}{\Longrightarrow}$ Supp D consists of the support of fibres of a. i.e. D = rational multiple of a fibre

$\Rightarrow$ for $m\gg 0$ & sufficiently divisible.

$$P_{nm}(S) = h^0(S, nmK_S) = h^0(S, mD) > 1 \quad ↯.$$

$\square$

Rmk these surfaces called bielliptic surfaces.

classified completely by Bagnera-De Franchis.

Case 5) $q = p_g = 1$. (Cannot occur)

let $\eta \in \mathrm{Pic}(S)$ be a non-trivial order 2 element.

then

$$\left.\begin{array}{l} h^0(\eta)+h^0(K_S-\eta) \geq \chi(\eta) \overset{R.R.}{=} \chi(\mathcal{O}_S)=1 \\ h^0(\eta)=0 \end{array}\right\} \Rightarrow h^0(K_S-\eta)\geq 1.$$

take $D \in |K_S-\eta|$ & $D' \in |K_S|$, then

$$2D,\ 2D' \in |2K_S| \overset{\chi(S)=0}{\Longrightarrow} 2D = 2D' \Rightarrow D = D'$$

$$\& \quad \eta = 0 \ \lightning.$$