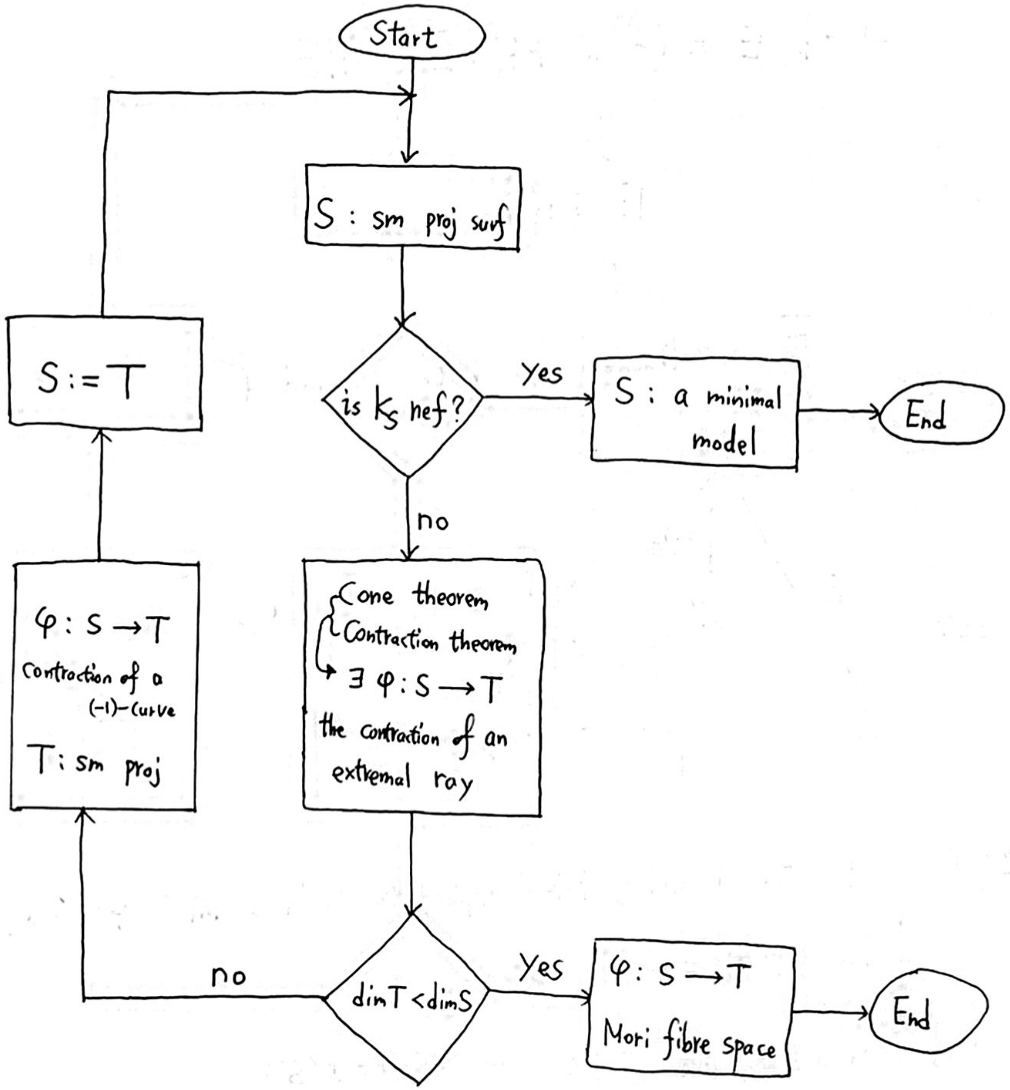
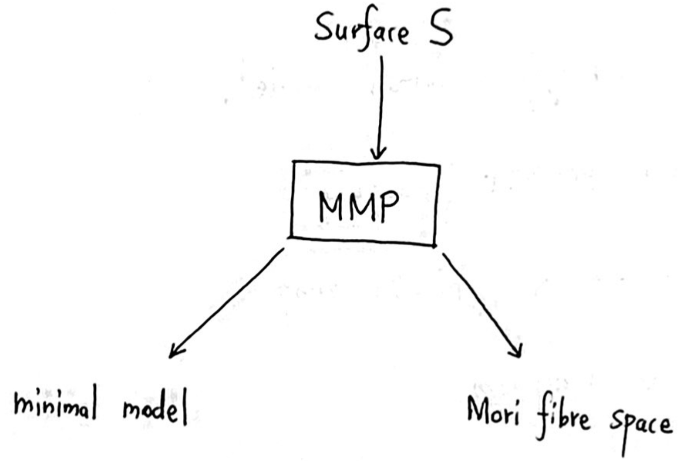
Minimal model program (for surfaces)



Summary



For convenience,

if K_S nef, then say S is <u>strongly minimal</u>.

Theorem (Uniqueness of the strongly minimal model)

let S be a strongly minimal model,

S' smooth projective surface

$f : S' \dashrightarrow S$ birat'l map

then f is a morphism.

In particular, if S, S' are both strong minimal models

& $f : S' \dashrightarrow S$ birat'l map

then it is an isom.

Pf. resolve the indeterminacies of f

$$\begin{tikzcd} X \arrow[d, "\varepsilon"'] \arrow[dr, "\tilde{f}"] & \\ S' \arrow[r, dashed, "f"'] & S \end{tikzcd}$$

WMA $\#\{$of blow-ups occuring in $\varepsilon\}$ is minimal

If ε is an isom, we are done.

If not, by ramification formula / blowups formula

$$K_X = \varepsilon^* K_{S'} + R' = \tilde{f}^* K_S + R \quad \text{with } R, R' \text{ both eff.}$$

ε not isom $\Rightarrow$ $\exists$ (-1)-curve $E \leq R'$, then

$$-1 = K_X E = (\tilde{f}^* K_S + R) E \geq RE$$

$$\Downarrow$$

$$E \leq R$$

$$\Downarrow$$

E contracted by both ε and $\tilde{f}$

$$\Longrightarrow \begin{tikzcd} & X \arrow[dl, "\varepsilon"'] \arrow[d, "\mathrm{contr}_E"] \arrow[dr, "\tilde{f}"] & \\ & X' \arrow[dl, "\varepsilon'"] \arrow[dr, "\tilde{f}'"'] & \\ S' \arrow[rr, dashed, "f"'] & & S \end{tikzcd} \qquad ↯$$

$\square$

Remark | If the result of MMP applied to S is a MFS $\varphi : S' \to T$ then $\kappa(S) = -\infty$

Indeed, S and S' are bir. eq. $\Rightarrow$ $\kappa(S) = \kappa(S')$

note either $S' = \mathbb{P}^2$ or $S' \to T$ is a scroll (i.e. $\mathbb{P}^1$-bundle)

In either case $\exists$ moving curves C on S' s.t. $C K_{S'} < 0$

$\Longrightarrow$ $P_n(S') = 0$ for all $n \in \mathbb{Z}_{>0}$ $\Rightarrow$ $\kappa(S') = \kappa(S) = -\infty$. $\square$

Fundamental theorem of the classification

THEOREM

S surface, then

(1) end result of MMP applied to S is a strong minimal model

$\Updownarrow$

$\kappa(S) \geq 0$

(2) end result of MMP applied to S is a Mori fibre space

$\Updownarrow$

$\kappa(S) = -\infty$

Remark that (1) and (2) are equiv.

We also know that

If $S \longrightarrow \boxed{\text{MMP}} \longrightarrow$ MFS, then $\kappa(S) = -\infty$.

(equivalently, if $\kappa(S) \geq 0$, then $S \to \boxed{\text{MMP}} \to$ Strongly minimal model

So above theorem reduces to the following theorem

Thm If end result of MMP applied to S is a strong minimal model, then $\kappa(S) \geq 0$.

Pf. (Argue by contradiction) Assume $\kappa(S) = -\infty$.

$\Rightarrow P_2(S) = h^0(2K_S) = 0$

Step 1 We have $q > 0$.

If $q = 0 \xrightarrow{\text{Castelnuovo rationality}} S$ rational surface ↯

Step 2 We claim that $q = 1$.

$$\underset{\substack{\uparrow \\ K_S \text{ nef}}}{0 \leq} \chi(\mathcal{O}_S) = 1 - q + p_g = 1 - q \Rightarrow 0 < q \leq 1$$

i.e. $q = 1$.

$\rightsquigarrow$ Albanese fibration $a : S \longrightarrow E$ elliptic curve

g : genus of the general fibre of a

K_S nef $\Rightarrow g > 0$ (in fact, if $g = 0$, then $S \cong E \times \mathbb{P}^1$)

<u>Step 3</u> assume $g=1$ $\qquad a: S \longrightarrow E$

by Canonical bundle formula,

$$K_S = a^*\left(K_E \otimes (R^1 a_* \mathcal{O}_S)^\vee\right) \otimes \mathcal{O}_S\left(\sum_{i=1}^{\ell}(m_i-1)F_i\right)$$

where $R^1 a_* \mathcal{O}_S$ line bundle of degree $= \chi(\mathcal{O}_S) = 0$ on E

$m_i F_i$: multiple fibres of a

$$\chi(\mathcal{O}_S) = (g(E)-1)(g-1) \Rightarrow \quad a \text{ is smooth}$$

$$\Downarrow \text{CBF}$$

$$K_S \equiv_{num} 0$$

<u>Fact</u>: S with K_S nef, $q=1$, $p_g=0$, $\kappa(S)\leq 0$

$a: S \longrightarrow E$ has genus 1

$\Rightarrow$ $\exists$ a morphism $\beta: S \to \mathbb{P}^1$ with connected fibres of genus 1.

CBF $\Rightarrow$ $\exists\, n \in \mathbb{Z}_{>0}$ & a line bundle M on $\mathbb{P}^1$ s.t.

$$nK_S = \beta^* M$$

$$K_S \equiv_{num} 0 \Rightarrow M \cong \mathcal{O}_{\mathbb{P}^1} \Rightarrow nK_S \simeq \mathcal{O}_S$$

$$\Downarrow$$

$$P_n(S) = 1$$

$$\Downarrow$$

$$\kappa(S) \geq 0 \quad \lightning$$

<u>Step 4</u> assume $g>1$

(~~Adjunction formula,~~) Consider the relative canonical sheaf

$$\omega_{S/E} = K_S \otimes a^*(K_E^\vee) \cong K_S$$

If $\deg a_* \omega_{S/E} > 0$, by R.-R. for vector bundles on curves

$$\chi(a_* \omega_{S/E}) = \deg(a_* \omega_{S/E}) > 0$$

$$\| $$

$$h^0(a_*\omega_{S/E}) - h^1(a_*\omega_{S/E})$$

$$\Rightarrow h^0(a_*\omega_{S/E}) > 0. \quad \Rightarrow p_g > 0 \quad \lightning.$$

$$\|$$

$$h^0(a_*\omega_S)$$

Hence $\deg a_* \omega_{S/E} = 0$ & $a: S \to E$ is isotrivial let F be a fibre of $\partial: S \to E$, $g(F) = g \geq 2$

then $\mathrm{Aut}(F)$ finite & $\pi_1(E) \curvearrowright F$

i.e. we have a homo.

$$\underset{\text{finite}}{\rho: \pi_1(E) \longrightarrow \mathrm{Aut}(F)}$$

$\Longrightarrow$ $\ker(\rho)$ has finite index in $\pi_1(E)$.

$\Longrightarrow$ $\exists$ an étale cover $C \xrightarrow{\theta} E$, here C elliptic

Consider the Cartesian diagram

$$\begin{tikzcd} S' \arrow[r, "\theta'"] \arrow[d, "\partial'"'] \arrow[dr, phantom, "\square"] & S \arrow[d, "\partial"] \\ C \arrow[r, "\theta"'] & E \end{tikzcd}$$

With $\partial': S' \to C$ also isotrivial, fibres $\cong F$

By def. $\pi_1(C)$ acts trivially on the fibres of ∂'

($i.e.\ \rho: \pi_1(C) \longrightarrow \mathrm{Aut}(F)$ injective)

$\Longrightarrow S' \cong C \times F$

$\Longrightarrow K_{S'} \cong pr_2^* K_F$ & $\kappa(S') = 1$

We claim that $\kappa(S) \geq 0$

$\theta': S' \longrightarrow S$ étale $\Rightarrow$ $nK_{S'} = \theta'^*(nK_S)$ for $\forall n \geq 1$.

$S \cong S'/G$ where G finite group of order m acting freely on S'

$\Longrightarrow$ $H^0(S, nK_S) = H^0(S', nK_{S'})^G$

Take $\forall\ 0 \neq t \in H^0(S', nK_{S'})$, consider the section

$t^G \in H^0(S', mnK_{S'})$ defined as

$$t^G(x) := \prod_{h \in G} \cancel{t(g(x))}\ t(h(x)) \quad \text{for } \forall x \in S'$$

then $t^G \neq 0$ & $t^G \in H^0(S', mnK_{S'})^G \simeq H^0(S, mnK_S)$

$\Rightarrow P_{mn}(S) > 0$

$\Rightarrow \kappa(S) \geq 0$ ↯.