Surfaces with non-nef canonical bundle

§0

Main theorem

S : smooth, irred, projective surface with K_S not nef

then $\exists$ irred ~~surface~~ variety T & a surjective morph $\varphi : S \twoheadrightarrow T$ s.t.

(called an extremal contraction)

① $\exists$ Curves on S Contracted to points by φ

($\rightsquigarrow$ φ not an isom.)

② If $C \subset S$ an irreducible curve Contracted by φ

then $K_S C < 0$

③ If C_1, C_2 two irred. curves on S both Contracted by φ

then $C_1 \underset{num}{\equiv} C_2 \in NS(S)$

④ If C_1, C_2 two irred. curves on S

$$\left.\begin{array}{l} \varphi(C_1) = pt \in T \\ C_1 \underset{num}{\equiv} C_2 \end{array}\right\} \Rightarrow C_2 \text{ also Contracted to a point by } \varphi$$

⑤ φ has Connected fibres, T is smooth & projective

The proof of main theorem based on following results.

§1

Rationality theorem

K_S not nef, A ample line bundle on S

define $r_A := \operatorname{Sup}\{t \in \mathbb{R}_{>0} : A + tK_S \text{ is nef}\}$

(Canonical threshold of A)

$\Longrightarrow$ $r_A \in \mathbb{Q}_{>0}$

Lemma 1 Suppose $\exists$ rational number $r_0 \geq r_A$ s.t. $k(A + r_0 K_S) \geq 0$ effective

for some $k \in \mathbb{Z}_{>0}$ i.e. $h^0(k(A + r_0 K_S)) > 0$

then $r_A \in \mathbb{Q}$

Pf of lem 1

Say $k(A + r_0 K_S) \underset{lin}{\sim} \sum_{i=1}^{n} d_i D_i$, where $\begin{cases} D_i \text{ distinct irred. curves} \\ d_i \in \mathbb{Z}_{>0} \end{cases}$

then $K_S = -\frac{1}{r_0} A + \frac{1}{k r_0} \sum_{i=1}^{n} d_i D_i \in \operatorname{Pic}(S)_{\mathbb{Q}}$

$$\Rightarrow \boxed{A + tK_S = \frac{r_0 - t}{r_0} A + \frac{t}{k r_0} \sum_i d_i D_i} \text{ for } \forall t \in \mathbb{Q}.$$

For $\forall$ irreducible curve $C \subset S$ different from the D_i

$\forall$ rational number t with $0 < t < r_0$, one has

$$(A + tK_S)C = \frac{r_0 - t}{r_0} AC + \frac{t}{kr_0} \sum_i d_i (D_i C) > 0$$

$\Rightarrow$ for $t \in (0, r_0) \cap \mathbb{Q}$,

$$A + tK_S \text{ nef} \Leftrightarrow (A + tK_S) D_i \geq 0 \text{ for } \forall 1 \leq i \leq n$$

$$\Updownarrow$$

$$\frac{r_0 - t}{r_0} AD_j + \frac{t}{kr_0} \sum_{i=1}^{n} d_i (D_i D_j) \geq 0$$

for $\forall 1 \leq j \leq n$

Thus

$$r_A = \min_{1 \leq j \leq n} \left\{ t_j \;\middle|\; \frac{r_0 - t_j}{r_0} AD_j + \frac{t_j}{kr_0} \sum_{i=1}^{n} d_i (D_i D_j) = 0 \right\}$$

taken over a finite set of rational numbers

$\Rightarrow r_A \in \mathbb{Q}$. $\square$

Lemma 2

If $r > 0$ irrational, then $\exists$ infinitely many pairs (u, v) of positive integers s.t.

$$0 < \frac{v}{u} - r < \frac{1}{3u}$$

Pf. given a positive irrational number r

$a_0 := \lfloor r \rfloor$ integer part

$a_1 := \lfloor \frac{1}{r - a_0} \rfloor = \lfloor r_1 \rfloor \qquad r_1 := \frac{1}{r - r_0}$

$\vdots$

$a_n = \lfloor r_n \rfloor \qquad a_{n+1} = \lfloor \frac{1}{r_n - a_n} \rfloor$

$\rightsquigarrow$ infinite continued fraction

$$r = a_0 + \cfrac{b_1}{a_1 + \cfrac{b_2}{a_2 + \cfrac{b_3}{\ddots}}}$$

where $C_0 = a_0 = \frac{A_0}{B_0}$

$C_1 = a_0 + \frac{b_1}{a_1} = \frac{A_1}{B_1}$

$C_2 = a_0 + \cfrac{b_1}{a_1 + \frac{b_2}{a_2}} = \frac{A_2}{B_2}$

$\vdots$

$C_n = \cdots = \frac{A_n}{B_n}$

Convention:

$$\boxed{\begin{array}{l} A_0 = a_0,\ B_0 = 1 \\ A_{-1} = 1,\ B_{-1} = 0 \end{array}}$$

three-term recurrence relation

$$A_n = a_n A_{n-1} + b_n A_{n-2}$$
$$B_n = a_n B_{n-1} + b_n B_{n-2} \qquad (n \geq 1)$$

with $c_1 < c_3 < \cdots < c_{2n+1} < \cdots < r < \cdots < c_{2n} < \cdots < c_4 < c_2$

&

$$|c_n - c_{n-1}| = \frac{1}{B_n B_{n-1}} \quad \text{for } \forall\, n \geq 1$$

If n is even, then

$$0 < c_n - r < \frac{1}{B_n B_{n-1}} < \frac{1}{3 B_n} \quad (\text{if } n \gg 0)$$

$\square$.

Pf of Rationality Theorem.

(argue by contradiction) assume r_A non-rational

For $\forall$ pair (x, y) of integers,

put

$$P(x,y) := \chi(\mathcal{O}_S(xA + yK_S))$$
$$= \chi(\mathcal{O}_S) + \frac{1}{2}(xA + yK_S)(xA + (y-1)K_S)$$

$\rightsquigarrow$ $P(x,y)$ is a polynomial of degree 2 in x, y & not identically 0.

- Now by lemma 2, $\exists$ ∞'ly pairs (u, v) of positive integers s.t. $0 < \frac{v}{u} - r_A < \frac{1}{3u}$.

- $P(ku, kv)$ is a quadratic polynomial in k, and

$P(ku, kv) \equiv 0 \iff$ the line $(vx - uy = 0) \subseteq (P(x,y) = 0)$ curve

(u,v) infinitely many pairs $\rightsquigarrow$ can choose (u_0, v_0) s.t.

$$P(ku_0, kv_0) \not\equiv 0 \text{ in } k$$
(not identically 0)

$\Downarrow$

$\exists\, k_0 \in \{1,2,3\}$ s.t.

$$P(k_0 u_0, k_0 v_0) \neq 0$$

Set

$$M = k_0(u_0 A + v_0 K_S)$$
$$= K_S + k_0 u_0 \underbrace{\left(A + \frac{k_0 v_0 - 1}{k_0 u_0} K_S\right)}_{\text{ample}}$$

$$0 < \frac{k_0 v_0 - 1}{k_0 u_0} = \frac{v_0}{u_0} - \frac{1}{k_0 u_0} \leq \frac{v_0}{u_0} - \frac{1}{3u_0} < r_A$$

by Kodaira vanishing

$$h^i(S, M) = 0 \quad \text{for } i > 0$$

$$\Rightarrow h^0(S, M) = \chi(S, M) = \chi(\mathcal{O}_S(k_0 u_0 A + k_0 v_0 K_S))$$
$$= P(k u_0, k v_0) \neq 0$$

$$\Rightarrow \quad r_0 := \frac{v_0}{u_0} > r_A \quad \text{s.t. } h^0(S, k_0 u_0 (A + r_0 K_S)) \neq 0$$

by lemma 1, $r_A \in \mathbb{Q}$ $\lightning$

$\square$

§2

Base-point freeness theorem

A ample divisor on S

Consider $L := A + r_A K_S \in \mathrm{Div}_{\mathbb{Q}}(S)$ nef

then $|mL|$ is base point free for $\left\{ \begin{array}{l} m \in \mathbb{Z}_{>0} \ \& \ m \gg 0 \\ \& \\ m \text{ suff divisible s.t. } m r_A \in \mathbb{Z} \end{array} \right.$

pf $\quad L$ nef $\Rightarrow L^2 \geq 0$

Case 1) $\quad L^2 > 0$

If L ample, then $|mL|$ very ample and hence bpf for $m \gg 0$.

So we may assume L nef but not ample.

$\Rightarrow \exists$ irreducible curve E s.t.

$$\begin{array}{c} L.E = 0 \\ \| \\ AE + r_A K_S E \end{array} > r K_S E \Rightarrow K_S E < 0 \qquad r_A = r$$

$$\left.\begin{array}{l} L^2 > 0 \\ LE = 0 \end{array}\right\} \xRightarrow{\text{Hodge index}} E^2 < 0 \longrightarrow E \ (-1)\text{-curve}$$

$\xrightarrow[(-1)\text{-curve}]{\text{Contracting}}$ $\exists$ a birational morphism

$$\mu : S \longrightarrow S' \leftarrow \text{Smooth}$$
$$E \longmapsto pt$$

$LE = 0 \Rightarrow L' := \mu_*(L)$ is a line bundle on S'

$\mu_* \mathcal{O}_S = \mathcal{O}_{S'} \Rightarrow \mu^*(L') = \mu^*\mu_*(L) \cong L \otimes \mu_* \mathcal{O}_S \cong L$

$$\left.\begin{array}{l} L'^2 = L^2 > 0 \\ L'C' = \mu^*(L').\mu^*(C') = L \cdot \mu^*(C') \geq 0 \\ \quad \text{for } \forall \text{ curve } C' \subset S' \end{array}\right\} \Rightarrow L' \text{ nef}$$

$K_S = \mu^* K_{S'} + E \Rightarrow L' = \mu_*(L) = \underbrace{\mu_*(A)}_{= A'} + r K_{S'} \qquad L = A + rK_S$

A' ample by Nakai-Moishezon

If L' ample, then $\exists\, m \in \mathbb{Z}$ s.t. $|mL'|$ bpf $\Rightarrow |mL|$ bpf

Otherwise L' nef but not ample & $L'^2 = L^2 > 0$

$\rightsquigarrow$ repeat above process, $\#\{(-1)\text{-curves}\} < +\infty \rightsquigarrow$ after finitely many steps, we arrive at a birational morphism $\varphi : S \to T$ where T is a smooth projective surface, obtained by blowing down finitely many (-1)-curves on S.

Moreover, we obtain an ample line bundle M on T s.t. $L = \varphi^* M$

$\Rightarrow$ M semi-ample $\Rightarrow$ L semi-ample

Case 2) $L \underset{num}{\equiv} 0$

$A + rK_S = L \equiv 0 \Rightarrow -K_S \equiv \frac{1}{r} A$ ample

$mL - K_S \underset{num}{\equiv} \frac{1}{r} A$ ample

Kodaira vanishing $\left[\begin{array}{l} X \text{ sm proj var}, \quad D \text{ nef \& big div} \\ \Rightarrow h^i(K_X + D) = 0 \text{ for } \forall i > 0 \end{array}\right]$

- $h^i(S, K_S - K_S) = 0$ for $\forall i > 0$

$\quad\quad \|$

$\quad h^i(S, \mathcal{O}_S)$ (i.e. $p_g(S) = q(S) = 0$)

- for $\forall$ integer m s.t. mL is a line bundle,

$$h^i(S, mL) = h^i(S, \underbrace{mL - K_S}_{\text{ample}} + K_S) = 0 \text{ for } \forall i > 0$$

$$\Rightarrow h^0(S, mL) = \chi(mL) = \chi(\mathcal{O}_S) + \frac{1}{2} mL(mL - K_S)$$
$$= \chi(\mathcal{O}_S)$$
$$= h^0(\mathcal{O}_S) = 1$$

$\left.\begin{array}{l} \Rightarrow mL \geq 0 \\ L \underset{num}{\equiv} 0 \end{array}\right\} \Rightarrow mL = 0$ In particular $|mL|$ no base pts

Case 3) $L^2 = 0$ & $L \underset{num}{\not\equiv} 0$

$\Rightarrow \exists$ irred curve $C \subset S$ s.t. $LC > 0$

Claim: $K_S L < 0$

$\ulcorner$ Indeed, take $k \gg 0$ s.t. $kA - C \underset{\text{effective}}{\geq} 0$

$$L \text{ nef} \Rightarrow L(kA) = \underbrace{L(kA - C)}_{\geq 0} + LC \geq LC > 0$$

$$\Downarrow$$

$$LA > 0$$

$$0 = L^2 = L(A + rK_S) = LA + rLK_S > rLK_S$$

$$\Downarrow$$

$$K_S L < 0$$

$\lrcorner$

For $\forall$ integer m $\qquad L = A + rK_S$

$$\rightsquigarrow \; mL - K_S = mL - \frac{1}{r}(-A+L) = \underbrace{\frac{1}{r}A}_{\text{ample}} + \underbrace{\frac{mr-1}{r}L}_{\text{nef } (m\gg 0)}$$

is ample, for $m \gg 0$

Kodaira vanishing $\Rightarrow$ $h^i(mL) = 0$ for $\forall i > 0$

$$\Downarrow$$

$$h^0(mL) = \chi(mL) = \chi(\mathcal{O}_S) + \frac{1}{2}mL(mL - K_S)$$
$$= \chi(\mathcal{O}_S) - \frac{1}{2}mL.K_S > 0$$

Write $|mL| = \underbrace{|M|}_{\text{movable part}} + \underbrace{F}_{\text{fixed part}}$ $\qquad (m \gg 0)$

$|mL|$ nef $\Rightarrow$ $|M|$ nef $\Rightarrow$

$$0 \leq M^2 \underset{F\geq 0}{\leq} M(M+F) = mL.M \underset{F\geq 0}{\leq} mL(M+F) = (mL)^2 = 0$$

$$\Longrightarrow M^2 = MF = F^2 = 0$$

$M^2 = 0 \Longrightarrow$ $|M|$ is composed with a pencil, that is, $\exists$ a curve D & a morph $f: S \longrightarrow D$ & a line bundle N on D s.t. $M = f^*N$.

- $MF = 0 \Rightarrow F \subseteq$ union of fibres of f — "f-vertical"
- $F^2 = 0 \xRightarrow{\text{Zariski Lemma}}$ F is a rational multiple of a fibre
- For $n \gg 0$, $|nN|$ on D without basepoints $\rightsquigarrow$ defines an embedding $D \hookrightarrow \mathbb{P}^N$

$$\Rightarrow |nmL| = \underline{f^*|nN|} + nF \quad \text{bpf}$$

$$\begin{tikzcd} S \arrow[d, "f"'] \arrow[dr, "|nmL|"] & \\ D \arrow[r, hook, "|nN|"] & \mathbb{P}^N \end{tikzcd}$$

Remark that $f: S \longrightarrow D$ is a $\mathbb{P}^1$-fibration, indeed,

Say F_η a general fibre of f

$$\left.\begin{array}{l} L.F_\eta = 0 \;\Longrightarrow\; K_S F_\eta < 0 \\ \quad \| \\ AF_\eta + rK_S F_\eta \qquad\quad F_\eta^2 = 0 \end{array}\right\} \Rightarrow F_\eta \cong \mathbb{P}^1$$

$\square$

§3 Boundedness of denominators

<u>Set-up</u> ~~Corollary~~ A ample line bundle

$$r := r_A := \sup\{t \in \mathbb{R}_{>0} \mid A + tK_S \text{ nef}\}$$

(canonical threshold of A)

by Rationality theorem, $r \in \mathbb{Q}_{>0}$

$L := A + r_A K_S$

<u>Corollary</u> $r = \dfrac{p}{q}$ with $\gcd(p,q) = 1$ & $q \in \{1,2,3\}$

<u>pf</u> Case 1) $L^2 > 0$

In this case, $\exists$ (-1)-curve E s.t.

$$\begin{aligned} &LE = 0 \\ &\ \ \| \\ &(A + rK_S)E \\ &\ \ \| \\ &AE - r \end{aligned}$$

$$\left.\begin{aligned} &\Rightarrow q(AE) = p \\ &\gcd(p,q) = 1 \end{aligned}\right\} \Rightarrow \underline{q = 1}$$

> $A = lH$ with $l > 0$.
> For any curve C
> $0 = LC = AC + rK_S C$
> $\quad = (l - kr)HC$
> $\Rightarrow r = \dfrac{l}{k}$, here $1 \leq k \leq 3$

Case 3) $L^2 = 0$ & $L \not\equiv_{num} 0$

In this case, $\exists$ irreducible curve F_η s.t. $F_\eta^2 = 0$, $K_S F_\eta = -2$

& $0 = L \cdot F_\eta = (A + rK_S)F_\eta = AF_\eta - 2r$

$\Rightarrow \underline{q = 2}$

Case 3) $L \equiv_{num} 0$

- <u>Assume $\rho > 1$</u>. $\Rightarrow$ $\exists$ ample divisor $A' \not\equiv_{num} A$

  $L' = A' + r'K_S$ nef

- If $L' \equiv_{num} 0$, then $rA' \equiv -rr'K_S \equiv r'A$ ↯

Hence for L' either Case 1) $\xrightarrow[LE=0]{\exists (-1)\text{-curve } E \subset S}$ $q = 1$

or Case 2) $\xrightarrow[LF=0]{\exists \text{ curve } F_\eta \text{ s.t. } K_S F_\eta = -2}$ ~~occurs~~ $q = 2$

- <u>Assume $\rho = 1$</u>

  Can choose an ample generator H of $\mathrm{Num}(S) = \mathrm{Div}(S)/\equiv_{num}$

  $\Rightarrow$ $-K_S = kH$ <u>claim</u>: $k \in \{1,2,3\}$. Indeed, if $k > 3$

for $x = 1,2,3$, we have $0 = h^0(K_S + xH) \underset{\substack{\uparrow \\ \text{Kodaira} \\ \text{Vanishing}}}{=} \chi(K_S + xH) := P(x)$ ← poly of deg 2

while having 3 roots $x = 1,2,3$ ↯

$K_S + xH = (x-k)H$ with $x - k < 0$

§4 Proof of extremal contraction theorem

Set-up: K_S not nef, A ample line bundle

then $\begin{cases} r_A := \sup\{t \in \mathbb{R}_{>0} \mid A + tK_S \text{ nef}\} \in \mathbb{Q}_{>0} \text{ by Rationality theorem} \\ L := A + r_A K_S \text{ nef} \overset{\text{bpf thm}}{\Longrightarrow} |mL| \text{ bpf for } m \gg 0 \text{ \& } m \text{ suff. divisible \& } mr_A \in \mathbb{Z} \end{cases}$

$\Rightarrow$ $|mL|$ defines a surjective morph.

$$\varphi_{|mL|} : S \longrightarrow T \subset \mathbb{P}^N$$

where T projective variety with $\dim T \leq 2$

Case $\dim T = 2$

We have $L^2 > 0$

$\Longrightarrow$ $\exists$ (-1)-curve E s.t. $LE = 0$

The contraction Contr_E of E is an extremal contraction

Case $\dim T = 1$

We can assume T smooth & $S \xrightarrow{\varphi} T$ has connected fibres (by Stein factorization) Say F a fibre of φ.

We have $FL = 0$

$$\underset{FA + rK_SF}{\overset{\|}{FL}} = 0 \Rightarrow \left.\begin{matrix} K_SF < 0 \\ F^2 = 0 \end{matrix}\right\} \Rightarrow K_SF = -2 \ \& \ F \cong \mathbb{P}^1 \text{ smooth}$$

If all fibres are irreducible, then $\varphi: S \to T$ is the desired extremal contraction.

Otherwise, say $F = \sum_{i=1}^{\ell} n_i F_i$ ~~an i~~ a reducible fibre

If $\ell = 1$, then $n_1 > 1$ & $F_1^2 = 0$ & $K_SF = -2$

$$K_SF = n_1 K_S F_1 \Downarrow n_1 = 2,\ K_SF_1 = -1 \ \lightning$$

must be even by adjunction formula.

If $\ell \geq 2$, then $F_i^2 < 0$ for $\forall\, 1 \leq i \leq \ell$.

$K_SF < 0 \Rightarrow K_SF_i < 0$ for some $1 \leq i \leq \ell$ $\Longrightarrow$ F_i (-1)-curve

$\rightsquigarrow$ the contraction of F_i is an extremal contraction.

Case $\dim T = 0$.

In this case $L \underset{num}{\equiv} 0$.

either $\exists$ (-1)-curve E on S, Contr_E is an extremal contraction

or no (-1)-curves but $\exists$ a morph. $f: S \to C$ over a