Pluricanonical Maps

Set-up: S minimal surface of general type

$\Rightarrow S$ contains a finite number of (-2)-curves.

C : the union of these curves

Connected Components $C^{(i)}$ of C are exceptional

$\Downarrow$

blowing down $C^{(i)}$, we get the normal singularity which is rational & of ADE-type.

$Z^{(i)} := \sum a_j C_j^{(i)}$ the fundamental cycle
$(a_j > 0)$

$S \longrightarrow S_{can}$ the blowing-down of C

$\uparrow$ possibly singular

pluricanonical maps

$$f_n := f_{|nK_S|} : S \dashrightarrow \mathbb{P}^N$$

$N = h^0(S, nK_S) - 1$

If f_n is a morphism, then it factors through S_{can}.

a map $k_n : S_{can} \longrightarrow \mathbb{P}^N$

Main Theorem S minimal surface of general type, then

① k_n is an embedding for $n \geq 5$

② k_4 is an embedding if $K_S^2 \geq 2$

③ k_3 is a morphism if $K_S^2 \geq 2$
an embedding if $K_S^2 \geq 3$

④ k_2 is a morphism if $K_S^2 \geq 5$.

If $K_S^2 \geq 10$, then k_2 is birational

$\Updownarrow$

S not fibred by genus 2 curves.

Consequences of this theorem

Prop 1 S surface of general type, then its canonical ring $R(S) := \bigoplus_{n \geq 0} H^0(S, nK_S)$ is a finitely generated noetherian ring.

Pf. If $\varepsilon : \tilde{X} \longrightarrow X$ is a blow-up of X at a point, then

$$\varepsilon^* : H^i(X, \mathcal{O}_X) \xrightarrow{\simeq} H^i(\tilde{X}, \mathcal{O}_{\tilde{X}}) \quad \text{for all } i \geq 0.$$

$$K_{\tilde{X}} = \varepsilon^* K_X \otimes \mathcal{O}_{\tilde{X}}(E)$$

$$\varepsilon^* : H^0(X, mK_X) \xrightarrow{\simeq} H^0(\tilde{X}, mK_{\tilde{X}}) \quad \text{for } \forall\, m \geq 1$$

So WMA S minimal. We have

$$R(S) = \bigoplus_{i\geq 0} H^0(\omega_S^{5i} \otimes \delta)$$

where $\delta = \mathcal{O}_S \oplus \omega_S \oplus \omega_S^2 \oplus \omega_S^3 \oplus \omega_S^4$

by Main Theorem, the map

$f_5 : S \longrightarrow \mathbb{P}^{k(5)}$ is a morphism

$\left(k(5) = \dim H^0(\omega_S^5) - 1 \right)$

$\Rightarrow \quad f_5^* \mathcal{O}_{\mathbb{P}}(1) = \omega_S^5$

$$\Rightarrow \quad \bigoplus_{i\geq 0} H^0(\omega_S^{5i} \otimes \delta) = \bigoplus_{i\geq 0} H^0(\mathcal{J}(i))$$

$\mathcal{J} := f_{5*}(\delta)$

put $A := \bigoplus_{j\geq 0} H^0(\mathcal{O}_{\mathbb{P}^{k(5)}}(j))$

then the module $\bigoplus_{i\geq 0} H^0(\mathcal{J}(i))$ is a finitely generated A-mod.

& A finitely generated ring

$\Rightarrow$ $R(S)$ also a finitely generated ring $\square$

abstract canonical model S_c

define the abstract canonical model

$$S_c := \operatorname{Proj} R(S)$$

since S surface of general type

$R(S) = \bigoplus_{d\geq 0} R_d(S)$

$\operatorname{tr.deg}_{\mathbb{C}} R(S) = 2$

$\Rightarrow$ S_c is an irreducible 2-dim'l projective variety.

The subring $R^{(n)}(S) := \bigoplus_{d\geq 0} R_{nd}(S)$ defines a proj var $X_c^{(n)} \cong X_c$.

$\cap$

$R(S)$

define $R^{[n]}(S)$ as the subring of $R^{(n)}(S)$ gen. by sections of $H^0(nK_S)$

$\rightsquigarrow$ the var. $S_c^{[n]}$ $\cancel{X_c^{[n]}}$ $= \operatorname{Proj}(R^{[n]}(S)) \subset \mathbb{P}^N$

$\uparrow$

the n-th canonical image

$\operatorname{Im}(f_n : S \longrightarrow \mathbb{P}^N)$

$\otimes$

$$\begin{tikzcd} & S_{can} \arrow[dr, "k_n"] & & \\ S \arrow[ur] \arrow[rr, "f_n"'] & & S_c^{[n]} \subset \mathbb{P}^N \end{tikzcd}$$

isom for $n \geq 5$

Serre's theorem $\Rightarrow$ $\exists$ integer d_0 s.t.

$$R_d^{[n]}(S) \cong H^0(S_c^{[n]}, \mathcal{O}_{S_c^{[n]}}(d)) \cong H^0(S_{can}, k_n^*\mathcal{O}_{\mathbb{P}}(d))$$
$$\cong H^0(S, \omega_S^{nd}) = R_d^{(n)}(S) \quad \text{for } d \geq d_0$$

$\Longrightarrow$ $S_c^{[n]} \cong S_c^{(n)} \cong S_c$ for $n \geq 5$.

If S minimal, then $S_{can} \cong \underset{\left(\substack{\text{abstract}\\ \text{canonical}\\ \text{model}}\right)}{S_c}$

Under this identification,

the map $k_n : S_{can} \longrightarrow S_c^{[n]}$, $S_{can} = S_c$

is just the map induced by the inclusion $R^{[n]}(S) \subset R(S)$

$(n \geq 5)$

Prop 2 | S surface of general type, integer $n \geq 2$ or $n < 0$
$\Rightarrow$ S minimal $\Leftrightarrow$ $H^1(nK_S) = 0$.

Pf. Recall Mumford's vanishing theorem

| X smooth projective surface
| L nef line bundle on X with $c_1^2(L) > 0$
| $\Rightarrow H^1(X, L^{-1}) = 0$

"$\Rightarrow$" Mumford's vanishing + Serre duality ✓

"$\Leftarrow$" If $n \geq 2$ & $\bar{S}$ obtained from S by blowing-up at least once,

then
$$h^1(nK_{\bar{S}}) \overset{R.R.}{=} P_n(\bar{S}) - \frac{n(n-1)}{2} K_{\bar{S}}^2 - \chi(\mathcal{O}_{\bar{S}})$$
$$> P_n(S) - \frac{n(n-1)}{2} K_S^2 - \chi(\mathcal{O}_S) = h^1(nK_S)$$

For $n < 0$, the argument is similar, use $h^2(nK_S)$ instead of P

Cor | If S minimal surface of general type
then $P_n(S) = \frac{n(n-1)}{2} K_S^2 + \chi(\mathcal{O}_S)$ for $\forall n \geq 2$.