Miyaoka-Yau inequality ($3c_2 \geq c_1^2$)

THEOREM (Miyaoka-Yau inequality)

For any surface of general type S, one has

$$3c_2 \geq c_1^2.$$

equivalently, $9\chi(\mathcal{O}_S) \geq K_S^2$.

To prove this inequality, we will give some lemmas for the preparation.

Lemma 1 (Bogomolov)
S surface
L line bundle on S with $h^0(L^\vee \otimes \Omega_S^1) \neq 0$
$\Rightarrow \exists$ constant c s.t. $h^0(L^m) \leq c \cdot m$ for $\forall m \geq 1$.

Pf. If for $\forall m \geq 1$, $h^0(L^m) \leq 1$, then take $c = 1$ ✓

So WMA $h^0(L^{m_0}) \geq 2$ for some $m_0 \geq 1$.

(baby case : $m_0 = 1$)

Say $s_1, s_2 \in H^0(L)$ two linearly independent sections

$0 \neq t \in H^0(L^\vee \otimes \Omega_S^1) \rightsquigarrow$ a nontrivial homo. $\mathcal{O}_S \xrightarrow{\cdot t} L^\vee \otimes \Omega_S^1$

$\rightsquigarrow$ a nontrivial homo. $h : L \longrightarrow \Omega_S^1$

$\Rightarrow h(s_1), h(s_2)$ linearly independent 1-forms on S

with $h(s_1) \wedge h(s_2) = 0 \in H^0(\Omega_S^2)$

$\overset{\text{CdF}}{\Longrightarrow} \exists$ smooth curve B of genus $g \geq 2$

a surjective morphism $f : S \longrightarrow B$ with connected fibres

such that $h(s_1), h(s_2) \in f^* H^0(\Omega_B^1)$.

If s_1 vanishes on a curve D, then this curve $\subseteq$ sum of some fibres of f

$\Rightarrow L = \mathcal{O}_S(D)$,

where $\forall$ component of $D \subseteq$ some fibre of f.

For $\forall$ ample divisor A, $n \gg 0$, F a fibre of f

$(D - nF)A < 0 \Rightarrow$ no effective divisors (homologously) of the form $m(D - nF)$ on S

(here $m > 0$ $\forall$ integer)

F_m : the divisor consisting of cm general fibres of f (hence smooth)

Consider the standard s.e.s.

$$0 \to \mathcal{O}_S(mD - F_m) \to \mathcal{O}_S(mD) \longrightarrow \mathcal{O}_{F_m}(mD) \longrightarrow 0$$

taking cohomology,

$$h^0(L^m) \leqslant h^0(\mathcal{O}_{F_m}(mD)) = cm \quad \text{for } \forall m \geqslant 1$$

(general case)

By the branched covering trick,

$\exists$ alg surface X

&

surj morph. $g : X \longrightarrow S$ s.t.

g^*L has 2 independent sections

$$h^0(\mathcal{H}om(L, \Omega_S^1)) \neq 0 \Rightarrow h^0(\mathcal{H}om(g^*L, \Omega_X^1)) \neq 0$$

$\Downarrow$ baby case

$h^0((g^*L)^m) \leqslant cm$ holds

$h^0(L^m) \leqslant cm$ for $\forall m \geqslant 1$. $\Leftarrow$

$h^0(L^{\otimes m}) \leqslant h^0((g^*L)^{\otimes m})$

$\square$

Lemma 2 S surface

$\mathcal{O}_S(D)$ line bundle

$\mathcal{F} \subset \Omega_S^1$ locally free, rank-2 subsheaf s.t.

$$\begin{cases} c_1(\mathcal{F}) \text{ nef line bundle} \\ h^0(\mathcal{H}om(\mathcal{O}_S(D), \mathcal{F})) \neq 0 \end{cases}$$

$$\Rightarrow c_1(\mathcal{F})D \leqslant \max(c_2(\mathcal{F}), 0).$$

Pf.

$$H^0(\mathcal{H}om(\mathcal{O}_S(D), \mathcal{F})) \cong H^0(\mathcal{F} \otimes \mathcal{O}_S(-D)) \neq 0$$

$\Rightarrow \exists$ effective divisor E on S s.t.

$\mathcal{F} \otimes \mathcal{O}_S(-D-E)$ admits a section with isolated zeros only.

$\Downarrow$

$$c_2(\mathcal{F} \otimes \mathcal{O}_S(-D-E)) \geqslant 0$$

$$\| $$

$$(D+E)^2 - c_1(\mathcal{F})(D+E) + c_2(\mathcal{F})$$

$$\Rightarrow c_1(\mathcal{F})D \leqslant (D+E)^2 + c_2(\mathcal{F}) - c_1(\mathcal{F})E$$

$c_1(\mathcal{F})$ nef $\Rightarrow c_1(\mathcal{F})E \geqslant 0$

If $(D+E)^2 \leqslant 0$, the desired result holds.

If $(D+E)^2 > 0$, by R.-R. to $\mathcal{O}_S(n(D+E))$

By the "branched covering trick"

$\exists$ surface X together with a surj morph. $f : X \to S$ (of deg k, say)

$$\begin{array}{ccc} \mathbb{P}(f^*\mathcal{F}) & \xrightarrow{\varphi} & \mathbb{P}(\mathcal{F}) \\ {\scriptstyle q}\downarrow & & \downarrow{\scriptstyle p} \\ X & \xrightarrow[f]{} & S \end{array}$$

$$f^*D = \sum_{i=1}^{n} D_i$$

D_i not necessarily eff.

$$\varphi^* G = \sum_{i=1}^{n} \left(H_{f^*\mathcal{F}} - q^* D_i \right)$$

For each i,

$$h^0 (\mathcal{H}om (\mathcal{O}_X (D_i), f^*\mathcal{F})) \neq 0$$

$f^*\mathcal{F} \subset \Omega^1_X$ subsheaf

$\Downarrow$

$$c_1 (f^*\mathcal{F}) D_i \leq \max (c_2 (f^*\mathcal{F}), 0)$$

$$f^*(c_1(\mathcal{F}) D) \leq \max (n c_2 (f^*\mathcal{F}), 0)$$

$$k\, c_1(\mathcal{F}) D \leq k \max (c_2 (\mathcal{F}), 0)$$

$$c_1(\mathcal{F}) D \leq \max (n c_2(\mathcal{F}), 0).$$

$\square$

Idea of proof :

assuming $c_1^2 > 3c_2$, a contradiction is obtained by showing that

- for suitable $\lambda \in \mathbb{Q}$, $n \in \mathbb{Z}$ with $n\lambda \in \mathbb{Z}$,

  Cohomology $H^i (S^n \Omega^1_S \otimes \mathcal{O}_S (n\lambda K_S))$ vanish for $i = 0, 2$, $n \gg 0$.

- $\chi (S^n \Omega^1_S \otimes \mathcal{O}_S (n\lambda K_S)) > 0$ for $n > n_0$.

Pf of MY ineq.

After blowing-up a point, c_1^2 drops by 1
c_2 increases by 1

$\rightsquigarrow$ We may assume S minimal.

Then we have $c_1^2(S) = K_S^2 > 0$ & $c_2(S) = e(S) > 0$.

We shall derive a contradiction from the assumption that

$$\alpha = \frac{c_2}{c_1^2} < \frac{1}{3} \quad (\text{i.e. } c_1^2 > 3c_2)$$

Let $\beta = \frac{1}{4}(1 - 3\alpha)$ & n the natural number s.t. $n(\alpha + \beta) \in \mathbb{Z}$.

$$h^0(n(D+E)) + \underbrace{h^2(n(D+E))}_{\substack{\| \\ h^0(K_S - n(D+E))}} \geq d n^2 \quad \text{for some } d>0 \ \& \ n \gg 0$$

$$h^0(\mathcal{H}om(\mathcal{O}_S(D), \mathcal{F})) \neq 0 \Rightarrow h^0(\mathcal{H}om(\mathcal{O}_S(D+E), \mathcal{F})) \neq 0$$

$$\Downarrow$$

$$h^0(n(D+E)) \leq cn \ \text{ for } \forall n \geq 1 \quad \overset{\text{lem1}}{\Longleftarrow} \quad h^0(\mathcal{H}om(\mathcal{O}_S(D+E), \Omega_S^1)) \neq 0$$

$$\Downarrow$$

$h^0(K_S - n(D+E)) \geq \frac{1}{2} d n^2$ for an infinite number of n's

$$\Downarrow$$

$c_1(\mathcal{F})(K_S - n(D+E)) \geq 0$ for an infinite number of n's

$c_1(\mathcal{F})(D+E) \leq \frac{1}{n} K_S c_1(\mathcal{F})$ for an infinite number of n's

$$\Rightarrow c_1(\mathcal{F})(D+E) \leq 0$$

i.e. $c_1(\mathcal{F}) D \leq -c_1(\mathcal{F}) E \leq 0$

$\square$

<u>lemma 3</u>

S surface

$\mathcal{O}_S(D)$ line bundle

$\mathcal{F} \subset \Omega_S^1$ locally free rank 2 subsheaf s.t.

$$\begin{cases} c_1(\mathcal{F}) \text{ nef line bundle} \\ h^0(\mathcal{H}om(\mathcal{O}_S(D), S^n\mathcal{F})) \neq 0 \end{cases}$$

$$\Rightarrow c_1(\mathcal{F}) D \leq \max(n c_2(\mathcal{F}), 0)$$

<u>Pf.</u>

$$\begin{array}{c} \mathbb{P}(\mathcal{F}) \\ p \downarrow \\ S \end{array} \qquad L := \mathcal{O}_{\mathbb{P}(\mathcal{F})}(1) \text{ tautological l.b.} = \mathcal{O}_{\mathbb{P}(\mathcal{F})}(H_{\mathcal{F}})$$

divisor class

<u>Fact</u>:

[For $\forall$ coherent sheaf $\mathcal{S}$ on S, $\exists$ canonical isom.

$$\mathcal{S} \otimes S^n\mathcal{F} \xrightarrow{\cong} p_*(p^*\mathcal{S} \otimes L^{\otimes n})$$]

For any divisor M on S,

$$H^0(\mathcal{O}_{\mathbb{P}(\mathcal{F})}(nH_{\mathcal{F}} + p^*M)) \cong H^0(\mathcal{O}_S(M) \otimes S^n\mathcal{F})$$

In our case, put $M = -D$, then $\exists$ effective div. G on $\mathbb{P}(\mathcal{F})$

s.t. $\mathcal{O}_{\mathbb{P}(\mathcal{F})}(G) = \mathcal{O}_{\mathbb{P}(\mathcal{F})}(nH_{\mathcal{F}} - p^*D)$

Consider the vector bundle

$$\mathcal{E}_n := S^n\Omega_S^1 \otimes \mathcal{O}_S(-n(\alpha+\beta)K_S)$$

<u>Claim</u> : $h^0(\mathcal{E}_n) = h^2(\mathcal{E}_n) = 0$, provided $n \gg 0$.

- take $\mathcal{F} = \Omega_S^1$, $D = n(\alpha+\beta)K_S$

  K_S nef & "$K_S \underset{\text{curve}}{C} = 0 \Leftrightarrow C$ is a (-2)-curve"

  by lemma 3, $h^0(\mathcal{E}_n) = 0$

- $h^2(\mathcal{E}_n) = h^2(S^n\Omega_S^1 \otimes \mathcal{O}_S(-n(\alpha+\beta)K_S))$

$$\overset{\text{Serre}}{=} h^0(S^nT_S \otimes \mathcal{O}_S((n(\alpha+\beta)+1)K_S))$$

(Ω_S^1 rank 2 v.b $\Rightarrow \Omega_S^1 \cong T_S \otimes \omega_S$)

$$= h^0(S^n\Omega_S^1 \otimes \mathcal{O}_S((\underline{n(\alpha+\beta-1)+1})K_S))$$

for $n \gg 0$, by lemma 3 $\|$ 0

Conclusion : $\chi(S^n\Omega_S^1 \otimes \mathcal{O}_S(-n(\alpha+\beta)K_S)) \leq 0$, for $n \gg 0$.

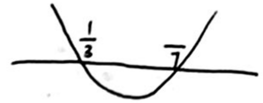
But, by Riemann-Roch, $\chi(S^n\Omega_S^1 \otimes \mathcal{O}_S(-n(\alpha+\beta)K_S))$ is a polynomial of degree 3 in n :

$$\chi = \frac{1}{6}\left[3(\alpha+\beta)^2 - 3(\alpha+\beta) - \alpha + 1\right]K_S^2 n^3 + O(n^2).$$

$$= \frac{K_S^2}{16}(3\alpha^2 - 22\alpha + 7) > 0 \quad \left(\alpha < \tfrac{1}{3} \text{ by assumption}\right)$$

$$\uparrow$$

$$\frac{3K_S^2}{16}\left(\alpha - \tfrac{1}{3}\right)(\alpha - 7)$$



To see this, consider the projective bundle over S

$$\mathbb{P}(\Omega_S^1) \xrightarrow{p} S$$

$$S^n\Omega_S^1 \otimes \mathcal{O}_S(-n(\alpha+\beta)K_S) \cong p_*\left(\underline{p^*\mathcal{O}_S(-n(\alpha+\beta)K_S) \otimes L^{\otimes n}}\right)$$

here $L = \mathcal{O}_{\mathbb{P}(\Omega_S^1)}(1)$ is the tautological line bundle on $\mathbb{P}(\Omega_S^1)$

<u>Fact</u> : $R^ip_*(p^*\mathcal{S} \otimes L^{\otimes n}) = 0$ for $\forall\, i > 0$, $\mathcal{S}$ coherent sheaf on S.

by asymptotic R.-R.

$$\chi(S^n\Omega_S^1 \otimes \mathcal{O}_S(-n(\alpha+\beta)K_S)) = \chi(p^*\mathcal{O}_S(-n(\alpha+\beta)K_S) \otimes L^{\otimes n})$$

$$\| $$

$$\frac{c_1^3(L \otimes p^*\mathcal{O}_S(-(\alpha+\beta)K_S))}{6} n^3 + k_2 n^2 + k_1 n + k_0$$

Calculate $c_1^3(L \otimes p^*\mathcal{O}_S(-(\alpha+\beta)k_S))$

By Grothendieck's relation, $r = \operatorname{rank}\Omega_S^1 = 2$ $\quad c_1(L) := \ell$

$$\sum_{i=0}^{r} (-1)^i p^* c_i(\Omega_S^1) \cdot \ell^{r-i} = 0 \in CH^r(\mathbb{P}(\Omega_S^1))$$

i.e. $\ell^2 + p^*(c_1(S))\ell + p^* c_2(S) = 0$

Since $\ell \cdot p^*(\text{natural generator of } H^4(S,\mathbb{Z})) = \text{natural generator of } H^6(\mathbb{P}(\Omega_S^1),\mathbb{Z})$

$$\Longrightarrow \quad \ell^3 = c_1^2 - c_2$$

We obtain

$$c_1^3(L \otimes p^*\mathcal{O}_S(-(\alpha+\beta)k_S)) = (\ell - (\alpha+\beta)p^*(-c_1(S)))^3$$

$$= \ell^3 - 3(\alpha+\beta)\ell^2 p^*(c_1(S)) + 3(\alpha+\beta)^2 \ell\, p^*(c_1^2(S))$$

$$= \frac{c_1^2}{16}(3\alpha^2 - 22\alpha + 7)$$

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then we get a contradiction!

$\square$