Some general properties of surfaces of general type

Prop 1 If $\exists$ an alg surface S & an ~~linear~~ algebraic system of eff div. of dim $\geqslant 1$ s.t. general member is (possibly singular) rational or elliptic curve,

then $\kappa(S) \leqslant 1$.

Pf. WMA this system to be irreducible & of dim 1.

$\Rightarrow$ $\exists$ smooth projective curve C with the product

$S \times C$ contains an irreducible 2-dim'l subvar X

$$\begin{tikzcd} X \arrow[r, phantom, "\subset"] \arrow[dr] & S \times C \arrow[d] \\ & S \end{tikzcd}$$

s.t. $X \twoheadrightarrow S$ & the desingularization $\tilde{X}$ of X is either a blowup ruled or an elliptic surf

$\Rightarrow \kappa(\tilde{X}) \leq 1$

$\Rightarrow \kappa(S) \leqslant 1$.

$\square$

THEOREM

S minimal surf of general type

then $K_S^2 = c_1^2(S) > 0$

Pf let H be a general hyperplane section of S.
(hence smooth)

Consider the s.e.s.

$$0 \to \mathcal{O}_S(nK_S - H) \to \mathcal{O}_S(nK_S) \longrightarrow \mathcal{O}_H(nK_S) \to 0$$

$$\left\{\begin{array}{l} S \text{ general type} \Rightarrow \exists \text{ constant } c > 0 \text{ s.t.} \\ \qquad h^0(\mathcal{O}_S(nK_S)) > cn^2 \text{ for } n \gg 0 \\ h^0(\mathcal{O}_H(nK_S)) \text{ grows linearly with } n \end{array}\right.$$

$\Longrightarrow \exists\, n_0 > 0$ s.t. $n_0 K_S - H \underset{\text{lin.}}{\sim} E \overset{\text{effective}}{\geqslant 0}$

$$\left.\begin{array}{l} S \text{ general type} \\ \text{minimal} \end{array}\right\} \Rightarrow K_S \text{ nef} \Rightarrow \underset{\substack{\| \\ n_0 K_S^2 - HK_S}}{K_S E} \geqslant 0$$

$$n_0^2 K_S^2 = n_0 K_S (H + E) \geqslant n_0 K_S H = H^2 + HE \geqslant H^2 > 0$$

$\Rightarrow K_S^2 > 0$.

Corollary | S minimal surf of general type

$C \subset S$ irreducible curve

then $K_S C \geq 0$

$K_S C = 0 \Leftrightarrow C$ is a (-2)-curve.

Pf. S general type $\Rightarrow nK_S \underset{\text{lin}}{\sim} \sum_{i=1}^{k} a_i D_i \overset{\text{eff}}{\geq} 0$

If $K_S C < 0$, then $\sum a_i D_i C < 0$

$\Downarrow$

C must be one of the D_i's

say $C = D_1$

$\Downarrow$

$$C\ (-1)\text{-curve} \Leftarrow \begin{cases} C(nK_S - a_1 C) \geq 0 \\ C^2 < 0 \end{cases}$$

$\lightning$

- If C (-2)-curve adjunction formula $\Rightarrow K_S C = 0$.
- If $\left.\begin{matrix} K_S C = 0 \\ K_S^2 > 0 \end{matrix}\right\} \xRightarrow{\text{Hodge index}} C^2 < 0 \xRightarrow[\text{formula}]{\text{adjunction}} C$ (-2)-curve $\square$

Prop2 | S surface of general type

then $c_2(S) = e(S) > 0$

Pf. Suppose that $e(S) < 0$

then $\exists$ étale cover $S' \xrightarrow{\pi} S$ s.t.

$$\begin{cases} p_g(S') \leq 2q(S') - 4 \\ e(S') < 0 \end{cases}$$

by a linear algebra fact,

$\exists$ holo. 1-forms $\omega_1, \omega_2 \in H^0(\Omega^1_{S'})$ s.t. $\omega_1 \wedge \omega_2 = 0$

$\xRightarrow{\text{CdF}}$ $\exists$ smooth curve B. & a surjective morphism $f: S' \to B$ with connected fibres

s.t. $\omega_1, \omega_2 \in f^* H^0(\Omega^1_B) \Rightarrow g(B) \geq 2$

$$\begin{array}{ccc} S' & \xrightarrow{\pi} & S \\ {\scriptstyle f}\downarrow & & \\ B & & \end{array}$$

If S' ruled, then so is S $\lightning$

hence a generic fibre F_η of f has genus $g \geq 1$

$\Rightarrow e(S') \geq e(B)\, e(F_\eta) = 4\chi(\mathcal{O}_B)\chi(\mathcal{O}_{F_\eta}) \geq 0$ $\lightning$

Prop3 S minimal surface of general type, then

① $\#\{(-2)\text{-curves on } S\} < +\infty$

they are independent $/\mathbb{Q}$ & $\#\{(-2)\text{-curves}\} \leq \rho(S)-1$

② On the subspace of $H_2(S,\mathbb{Q})$, gen. by the (-2)-curves, the intersection form is negative definite.

Pf. If $C_1, \cdots, C_\ell$ are (-2)-curves on S, s.t.

$$\sum_{i=1}^{k} \lambda_i C_i = \sum_{j=k+1}^{\ell} \lambda_j C_j \in H_2(S,\mathbb{Q})$$

for some $1 \leq k \leq \ell$. $\lambda_i \geq 0$ for $\forall i$.

then

$$\left(\sum_{i=1}^{k} \lambda_i C_i\right)^2 = \left(\sum_{i=1}^{k} \lambda_i C_i\right)\left(\sum_{j=k+1}^{\ell} \lambda_j C_j\right) \geq 0$$

Also

$$\left.\begin{array}{l} K_S^2 > 0 \\ K_S\left(\sum_{i=1}^{k} \lambda_i C_i\right) = 0 \end{array}\right\} \xrightarrow{\text{Hodge index}} \sum_{i=1}^{k} \lambda_i C_i = 0 \in H_2(S,\mathbb{Q})$$

$$\Downarrow$$

$$\lambda_i = 0 \quad \forall\, 1 \leq i \leq \ell.$$

Since each rat'l homology class contains at most one (-2)-curve.

$$\#\{(-2)\text{-curves}\} \leq \rho(S)$$

$K_S H > 0$ for a hyperplane section $H \Rightarrow H$ not homologous to a sum $\sum \lambda_i C_i$

$$\Downarrow$$

$$\#\{(-2)\text{-curves}\} \leq \rho(S)-1.$$

② follows from $K_S^2 > 0$ & Hodge index theorem

$\square$