Bielliptic surfaces

Def. A surface with $\kappa = 0$, $q = 1$, $p_g = 0$ is called a bielliptic surface.

Bagnera-de Franchis theorem

The bielliptic surface S is of the form $E \times F/G$, where

E, F smooth elliptic curves

G a group of translations of E acting on F

with

(1) $G = \mathbb{Z}/2$ acting on F as $\begin{array}{c} F \longrightarrow F \\ x \longmapsto -x \end{array}$

$2k_S \sim 0$

(2) $G = \mathbb{Z}/2 \oplus \mathbb{Z}/2$ acting on F by $x \mapsto -x$ &

$x \longmapsto x + \varepsilon$, where ε is a nontrivial torsion 2 point on F

(3) $F = F_i = \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}i$, $G = \mathbb{Z}/4$ acting on F by $x \mapsto ix$

$4k_S \sim 0$

(4) $F = F_i = \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}i$ $G = \mathbb{Z}/4 \oplus \mathbb{Z}/2$ acting on F by

$x \longmapsto ix$ & $x \longmapsto x + \frac{1+i}{2}$

(5) $F = F_\rho = \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\rho$ (ρ nontrivial cubic root of 1)

$3k_S \sim 0$ $G = \mathbb{Z}/3$ acting on F by $x \longmapsto \rho x$

(6) $F = F_\rho = \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\rho$. $G = \mathbb{Z}/3 \oplus \mathbb{Z}/3$ acting on F by

$x \longmapsto \rho x$ & $x \longmapsto x + \frac{1-\rho}{3}$

(7) $F = F_\rho = \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\rho$, $G = \mathbb{Z}/6$ acting on F by $x \mapsto -\rho x$.

$6k_S \sim 0$

We start with the following lemma

lemma 1

A minimal bielliptic surface S is the quotient of the product of two elliptic curves via a free group action.

More precisely, $\exists$ Commutative diagram

$$\begin{tikzcd} S' \arrow[r, "g"] \arrow[d, "\alpha'"'] \arrow[dr, phantom, "\circlearrowright"] & S \arrow[d, "\alpha"] \\ E \arrow[r, "f"'] & A \end{tikzcd}$$

where $\alpha : S \longrightarrow A = \mathrm{Alb}\,S$ Albanese fibration (isotrivial elliptic fibration)

$S' = E \times F$ product of two smooth elliptic curves

$\alpha' = pr_1$

f, g are étale, induced by a free action of an abelian group G

Pf. $\alpha : S \longrightarrow A = \mathrm{Alb} S$ Albanese fibration

elliptic curve

$\deg \alpha_* \omega_{S/A} = \chi(\mathcal{O}_S) - (g(A)-1)(g(F)-1) = 0 \Rightarrow \alpha$ smooth, isotrivial

$\Rightarrow \exists$ étale cover $C \longrightarrow A$ s.t.

$$\begin{tikzcd} C \times F \cong S' \arrow[r] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & S \arrow[d, "\alpha"] \\ C \arrow[r] & A \end{tikzcd}$$

$$\left.\begin{array}{l} \left.\begin{array}{l} C \text{ elliptic curve} \\ \text{if } g(F) \geq 2 \end{array}\right\} \Rightarrow \kappa(S') = 1 \\ \text{Can prove that } \kappa(S) = \kappa(S') \end{array}\right\} \Rightarrow \text{↯} \Rightarrow \underline{g(F) = 1}.$$

$p_g = 0 \Rightarrow$ a multiple of K_S is trivial $\Rightarrow K_S \in \mathrm{Num}\, S$

non-trivial torsion point

By canonical bundle formula,

$$K_S = \alpha^*\left(K_A \otimes (R^1\alpha_*\mathcal{O}_S)^\vee\right) \otimes \mathcal{O}_S\left(\sum_{i=1}^{k}(m_i - 1)F_i\right)$$

$\Longrightarrow$ α has no multiple fibres.

$(R^1\alpha_*\mathcal{O}_S)^\vee$ torsion line bundle on A

Remark that α is smooth, then we have a morphism

$$\mu : A \longrightarrow \mathcal{M}_1$$
$$x \longmapsto F_x$$

where $\mathcal{M}_1 \simeq \mathbb{A}^1$ is the moduli space of curves of genus 1.

$$\left.\begin{array}{l} A \text{ projective} \\ \mu : A \longrightarrow \mathcal{M}_1 \cong \mathbb{A}^1 \end{array}\right\} \Rightarrow \mu \text{ constant} \Rightarrow \text{the elliptic fibration}$$

$\alpha : S \longrightarrow A$

is isotrivial with fibre F

(as noted before)

$\Rightarrow$ $\exists$ étale cover $E \longrightarrow A$ s.t.

$$\begin{tikzcd} E \times F = S' \arrow[r, "g"] \arrow[d] \arrow[dr, phantom, "\lrcorner"] & S \arrow[d, "\alpha"] \\ E \arrow[r, "f"'] & A \end{tikzcd}$$

f, g are étale induced by the action of a finite group G acting freely on S' and E

$E/G \cong A \Rightarrow G$ is a finite group of translations of E

In fact, if $E = \mathbb{C}/\Lambda$, $A = \mathbb{C}/\Lambda'$

then $\Lambda \subseteq \Lambda'$ & $G = \Lambda'/\Lambda$

$\Rightarrow$ G abelian.

$\square$

lemma 2 In the same set-up as above,

$\exists$ smooth elliptic curve E'

$\exists$ étale morphism $h : E' \longrightarrow E$

$\exists$ a finite group G' acting on E' & F such that

- $E'/G' \simeq E/G = A$
- $E' \times F/G \simeq E \times F/G = S$

By lemma 2, up to replacing E by E', we may assume G is a finite group of translations of E acting also on F, & diagonally on $S' = E \times F$.

Put $C = F/G \Rightarrow g(C) \leq 1$

Claim: $g(C) = 0$.

Indeed, ~~if $g(C)=1$~~, ~~then the morphism $S \longrightarrow C$ factors through a~~

$$\underset{\underset{1}{\|}}{q(S)} = g(F/G) + \underset{\underset{\underset{1}{\|}}{g(A)}}{\underset{\|}{g(E/G)}} \Rightarrow g(C) = 0$$

Fact E smooth elliptic curve (viewed as 1-dim A.V.)

j : j-invariant of E

$\mathrm{Aut}_0(E)$: group of automorphisms of E as abelian var.

$\Rightarrow \mathrm{Aut}_0(E)$ is a finite group.

More precisely,

(1) it is $\mathbb{Z}/2$, generated by $x \longmapsto -x$ if $j \neq 0, 1728$

(2) $\mathbb{Z}/4$, generated by $x \longmapsto ix$ if $j = 1728$. (i.e. if $E = \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}i$)

(3) $\mathbb{Z}/6$, generated by $x \longmapsto -\rho x$ if $j = 0$. (i.e if $E = \mathbb{C}/\mathbb{Z}\oplus\mathbb{Z}\rho$) ($\rho$: nontrivial cubic root of unity)

Moreover,

$$\mathrm{Aut}(E) = \underset{\substack{\uparrow \\ \text{translations}}}{E} \ltimes \mathrm{Aut}_0(E)$$

proof of Bagnera-de Franchis theorem

By lemma 1, $S = E \times F / G$ $\begin{cases} E, F \text{ elliptic curves} \\ G \subset E \text{ finite abelian group of translantions} \\ G \subset \mathrm{Aut}(F) \text{ s.t. } F/G \simeq \mathbb{P}^1 \end{cases}$

$G = T \times H \Leftarrow$

$\begin{cases} T : \text{finite subgroup of translations of } F \\ H \subset \mathrm{Aut}_0(F) \text{ subgroup} \end{cases}$

$F/G \cong \mathbb{P}^1 \Rightarrow H \neq 0 \Rightarrow H = \mathbb{Z}_p$ with $p = 2, 3, 4, 6$

G abelian $\Rightarrow$ elements of T commute with those of H.
[i.e. they are translations by the H-fixed points]

$\Rightarrow$

(1) if $H = \mathbb{Z}/2$ (generated by $x \mapsto -x$)

fixed points are $\underbrace{\text{order 2 points on } F}_{F[2]}$

(2) if $H = \mathbb{Z}/4$ (gen. by $x \mapsto ix$) & $F = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}i$

then fixed points are 0 & $\frac{1+i}{2}$

(3) if $H = \mathbb{Z}/3$. (gen by $x \mapsto \rho x$) & $F = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\rho$

then fixed points are 0 & $\pm \frac{1-\rho}{3}$.

(4) if $H = \mathbb{Z}/6$. (gen. by $x \mapsto -\rho x$) & $F = \mathbb{C}/\mathbb{Z} \oplus \mathbb{Z}\rho$

then only fixed point is 0.

note that $G = T \times H$ is a subgroup of E

$\Rightarrow$ $G = F[2] \times \mathbb{Z}/2$ is impossible.

As for the triviality of pluricanonical divisor.

let $0 \neq \omega \in H^0(\Omega^2_{E \times F})$

then the minimum n s.t. nK_S trivial
$\parallel$
minimum n s.t. G acts trivially on ω^n.

$\square$