Complex tori & abelian varieties

Complex tori

$\Lambda \subset \mathbb{C}^g$ a lattice, that is a discrete subgroup of $\mathbb{C}^g$

with $\Lambda \otimes_{\mathbb{Z}} \mathbb{R} \cong \mathbb{R}^{2g}$

(i.e. free abelian group of rank $2g$. $\cong \mathbb{Z}^{2g}$)

quotient space $T := \mathbb{C}^g/\Lambda$, endowed with

$\boxed{\text{quotient topology}}$.

$$\pi: \mathbb{C}^g \longrightarrow T := \mathbb{C}^g/\Lambda$$
$$z \longmapsto z + \Lambda$$

$\hookrightarrow U \subset T$ open subset $\overset{\text{def}}{\iff} \pi^{-1}(U) \overset{\text{open}}{\subset} \mathbb{C}^g$

<u>Fact 1</u>: T is compact

Step 1 Use fundamental domain of lattices to construct finite cover

$\Lambda \cong \mathbb{Z}^{2g} \Rightarrow$ take a $\mathbb{R}$-basis of Λ $\{v_1, \dots, v_{2g}\}$

(i.e. $\Lambda \cong \mathbb{Z}v_1 \oplus \dots \oplus \mathbb{Z}v_{2g}$)

Consider the fundamental domain

$$F := \left\{ a_1 v_1 + \dots + a_{2g} v_{2g} \;\middle|\; 0 \le a_i < 1 \ (\text{for } \forall\ 1 \le i \le 2g) \right\}$$

then F is a $\underbrace{\text{bounded closed}}_{\text{(Compact set)}}$ subset of $\mathbb{R}^{2g} \cong \mathbb{C}^g$ and

F has the property that

- $\mathbb{C}^g = \bigcup_{v \in \Lambda} (F + v)$

- for two distinct points $v_1, v_2 \in \Lambda$.

  $(F + v_1) \cap (F + v_2)$ no interior points,

  doesn't affect the covering

Step 2 A continuous map maps compact set to compact set.

quotient map $\pi: \mathbb{C}^g \longrightarrow T$ continuous

$$z \longmapsto z + \Lambda := [z]$$

F bounded closed set in $\mathbb{C}^g \Rightarrow F$ compact

$\pi(F) = T \Rightarrow T$ compact.

<u>Fact 2</u>: T is an analytic manifold.

- Housdorff

For $\forall\ [z_1] \neq [z_2] \in T. \Rightarrow z_1 - z_2 \notin \overset{\text{discrete}}{\Lambda}$

$\Rightarrow \exists$ neighborhoods $z_1 \in U$, $z_2 \in V$ s.t.

$(U + \Lambda) \cap (V + \Lambda) = \emptyset$

$\overset{T}{\underset{\text{Housdorff}}{\Leftarrow}}$ $\pi(U)$ & $\pi(V)$ are disjoint open nbhds of $[z_1]$, $[z_2]$ resp. $\Longleftarrow$

Second-countable

$\mathbb{C}^g$ second-countable (i.e. admits countable basis $\mathcal{B}$)
T quotient topology via $\pi : \mathbb{C}^g \to T$
$\Rightarrow$ T second-countable ($T(\mathcal{B})$ countable basis)

local analytic homeomorph. to $\mathbb{C}^g$

For $\forall\, [z] = z + \Lambda \in T$, take a nbhd $U \subset \mathbb{C}^g$ of z s.t.

key!!! $\boxed{U \cap (U + v) = \phi \quad \text{for } \forall\, 0 \neq v \in \Lambda}$

(this is possible, since Λ discrete, $d := \text{distance}(z, z+v) > 0$
$\rightsquigarrow$ can take U as a small open ball $B(z, \varepsilon)$ around z with radius $\varepsilon < \frac{1}{2} d$)

$\Rightarrow$ the restriction map

$$\pi|_U : U \longrightarrow \pi(U) \text{ is a homeomorphism}$$

- $\pi|_U$ continuous by def. of quotient topology
- $\pi|_U$ injectivity : if $\pi(u_1) = \pi(u_2)$ $(u_1, u_2 \in U)$ $\Rightarrow$ $u_1 - u_2 \in \Lambda$
  $U \cap (U+v) = \phi$ for $\forall\, 0 \neq v \in \Lambda$
  $u_1, u_2 \in U$
  $\Rightarrow u_1 = u_2$

- $\pi|_U$ surjectivity :
  for $\forall\, [w] \in \pi(U) \Rightarrow \exists\, w \in U + v$ for some $v \in \Lambda$
  $U \cap (U+v) = \phi$ for $\forall\, 0 \neq v \in \Lambda$
  $v = 0$
  i.e. $w \in U$

- Inverse also continuous. $\pi|_U : U \longrightarrow \pi(U)$ $\quad \pi^{-1}|_U : \pi(U) \to U$, $U \supset V$
  if $V \subseteq U$ is open, then $V + \Lambda$ open in $\mathbb{C}^g$
  $$\Rightarrow \pi(V) = \pi(U) \cap \pi(V + \Lambda)$$
  open in T

Finally, $\pi|_U$ is an analytic isom.

$\pi : \mathbb{C}^g \longrightarrow T$ holomorphic submersion
(i.e. $(d\pi)_z : T_z \mathbb{C}^g \xrightarrow{\cong} T_{[z]} T$ for $\forall\, z \in \mathbb{C}^g$)

- $\pi|_U$ holomorphic.

- $(\pi|_U)^{-1} : W := \pi(U) \longrightarrow U$ inverse homeomorphic

$$\begin{array}{ccc} W := \pi(U) & \longrightarrow & U \\ \uparrow \pi & & \downarrow \varphi \text{ holomorphic function} \\ U & & \mathbb{C} \end{array}$$

$(\pi|_U^{-1})^*(\varphi)$ holomorphic $\Leftrightarrow$ $\varphi = (\pi|_U)^* (\pi|_U^{-1})^*(\varphi)$ holomorphic

$\pi : \mathbb{C}^g \longrightarrow T$

T has an abelian group structure naturally from additive group of $\mathbb{C}^g$ ($\Lambda \subset \mathbb{C}^g$ subgroup under addition)

Moreover, addition & inversion on T are holomorphic

$\rightsquigarrow$ T complex abelian Lie group.

<u>Abelian varieties</u>

<u>Def</u> We say that a complex torus $\mathbb{C}^g/\Lambda$ is an <u>abelian variety</u> if it can be embedded into some projective space $\mathbb{P}^N$ as a complex submanifold. By Chow's theorem, it is a projective algebraic variety.

Set-up:

$V \cong \mathbb{C}^g$ g-dim'l $\mathbb{C}$-vector space

Λ lattice in V

$T := V/\Lambda$ complex torus, $\pi : V \longrightarrow T$ quotient map

For $\forall p \in T$, $T_p(T) \underset{\text{via translation}}{\cong} T_0(T) \cong V$

$\overset{T \text{ group}}{\text{Var.}} \Rightarrow$ tangent sheaf of $T \overset{\text{can. isom.}}{\cong}$ free sheaf $V \otimes_{\mathbb{C}} \mathcal{O}_T$

dually, cotangent sheaf of $T \overset{\text{can. isom.}}{\cong} V^* \otimes_{\mathbb{C}} \mathcal{O}_T$

<u>Rmk</u>:

☆ In particular, $\exists$ an isom. $\delta : V^* \xrightarrow{\sim} H^0(T, \Omega^1_T)$ given explicitly as follows:

a form $x^* \in V^*$ defines a ~~function~~ holo 1-form ω_{x^*} on V ~~satisfying~~

~~$x^*(v+\gamma) = x^*(v) + \text{constant}$ for $\forall v \in V, \gamma \in \Lambda$~~

$$V^* \longrightarrow H^0(V, \Omega^1_V)$$
$$x^* \longmapsto \omega_{x^*} \quad \left(\begin{array}{l} \text{for } \forall v \in V, \ a \in T_vV \cong V \\ \underbrace{\omega_{x^*}(v)}_{\uparrow \text{ 1-form at } v \in V} (a) := x^*(a) \end{array} \right)$$

It is an isom.

- $\pi : V \longrightarrow T$ surjective $\Rightarrow$ $\pi^* : H^0(T, \Omega^1_T) \underset{\text{injective}}{\hookrightarrow} H^0(V, \Omega^1_V)$

Claim:

- <u>All holomorphic 1-form on T are Λ-translation invariant in $H^0(\Omega^1_V)$</u>

Indeed, for $\lambda \in \Lambda$, let $\tau_\lambda : V \longrightarrow V$, $v \longmapsto v+\lambda$ be the translation-by-λ map

$$\begin{tikzcd} V \arrow[r, "\pi"] \arrow[d, "\tau_\lambda"'] & V/\Lambda := T \\ V \arrow[ur, "\pi"'] \arrow[u, phantom, "\circlearrowleft"] \end{tikzcd} \Rightarrow \boxed{\pi \circ \tau_\lambda = \pi} \Rightarrow \text{for } \forall \omega \in H^0(\Omega^1_T)$$

$\boxed{\tau_\lambda^* \pi^* \omega = \pi^* \omega} \Rightarrow \pi^* \omega$ Λ-transl. inv.

Conversely, if $\eta \in H^0(V, \Omega_V^1)$ is Λ-translation invariant,

then $\exists!\ \omega \in H^0(\Omega_T^1)$ s.t. $\pi^*\omega = \eta$:

for $t = v + \Lambda \in T$, define

$$\omega(t) := \eta(v) \circ (d\pi_v)^{-1}$$

$v \in V$, $\pi : V \to T$ $\qquad d\pi_v : T_v(V) \longrightarrow T_t(T)$

$$T_t(T) \xrightarrow{(d\pi_v)^{-1}} T_v(V) \xrightarrow{\eta(v)} \mathbb{C}$$

η Λ-inv. $\Rightarrow$ $\omega(x)$ is independent of v. & ω holo.

Claim:

• $\underline{\eta \in H^0(\Omega_V^1)\ \ \Lambda\text{-translation inv.} \Leftrightarrow \eta \in V^*}$

"$\Rightarrow$" In complex coordinates $z_1, \cdots, z_n$ on V, $\eta = \sum_{i=1}^{n} f_i(z)\, dz_i$ ← holo.

η Λ-translation inv. $\Rightarrow$ $f_i(z+\lambda) = f_i(z)$ for $\forall \lambda \in \Lambda$

for $\forall v \in V$, $v = g + \lambda$, $g \in F$, $\lambda \in \Lambda$; $f_i(v) = f_i(g)$ $\Leftarrow$
$\left\{ \begin{array}{l} f_i \text{ periodic w.r.t. } \Lambda \\ \exists \text{ compact fundamental domain } F \subset V \end{array} \right.$

(closed bounded subset s.t. $\forall v \in V$ $v = g + \lambda$ for $g \in F$, $\lambda \in \Lambda$) ← unique rep.

$\left.\begin{array}{l} F \text{ compact} \\ f_i|_F \text{ continuous function} \end{array}\right\} \Rightarrow f_i|_F$ bounded on F

$\Downarrow$

$\left.\begin{array}{l} f_i \text{ bounded on whole of } V \ (\text{holomorphic}) \\ V \text{ connected} \end{array}\right\} \overset{\text{Liouville's thm}}{\Rightarrow} f_i$ constant

$\eta = \sum c_i dz_i \in V^*$

"$\Leftarrow$" for $x^* \in V^*$, $\eta = \omega_{x^*} \in H^0(\Omega_V^1)$ satisfies

$$\boxed{\begin{array}{l} \text{for } v \in V,\ a \in T_v V \cong V \\ \omega_{x^*}(v)(a) = x^* a \end{array}}$$

$\tau_\lambda : V \longrightarrow V$

$d\tau_\lambda(v) : T_v V \to T_{v+\lambda} V$, $\ni a$

$$(\tau_\lambda^*\eta)(v)(a) = \eta(v+\lambda)\big(d\tau_\lambda(v)(a)\big)$$
$$= x^*(a)$$
$$\overset{\text{def}}{=} \eta(v)(a) \quad \Rightarrow \eta \text{ translation-invariant.}$$

• $\pi^* : H^0(\Omega_T^1) \xrightarrow{\sim} \{\eta \in H^0(\Omega_V^1) \mid \eta\ \Lambda\text{-translation inv.}\} = V^*$

Lemma $T_1 = V_1/\Lambda_1$, $T_2 = V_2/\Lambda_2$ two complex tori

$\theta : T_1 \longrightarrow T_2$ a morphism.

$\Rightarrow$ θ factors as

$$\begin{tikzcd} T_1 \arrow[r, "\tau"', "\text{Translation}"] \arrow[rr, bend left, "\theta"] & T_1 \arrow[r, "\tilde\theta"] & T_2 \end{tikzcd}$$

$\tilde\theta$: group homo.

Where $\tilde\theta$ induced by a linear map $\bar\theta : V_1 \longrightarrow V_2$

with $\bar\theta(\Lambda_1) \subset \Lambda_2$ $\qquad \Lambda_1 \longmapsto \bar\theta(\Lambda_1) \subset \Lambda_2$

In particular, $\tilde\theta$ determined by $\theta^* : H^0(\Omega_{T_2}^1) \longrightarrow H^0(\Omega_{T_1}^1)$

$\underline{\underline{\text{pf}}}$.

$$\begin{tikzcd} V_1 \arrow[r, "\bar\theta"] \arrow[d, "\pi_1"'] & V_2 \arrow[d, "\pi_2"] \\ T_1 \arrow[r, "\theta"] & T_2 \end{tikzcd}$$

π_1, π_2 are universal cover of T_1, T_2, resp. By the ~~universal~~ lifting property (w.r.t. ~~of~~ the cover π_2) ($\theta_* \pi_{1*}(\pi_1(V_1)) \subset \pi_{2*}(\pi_1(V_2))$)

$\exists$ a morph. $\bar\theta : V_1 \longrightarrow V_2$ s.t. $\bar\theta(x+\lambda) - \bar\theta(x) \in \Lambda_2$

$(x \in V_1, \lambda \in \Lambda_1)$

$\Downarrow$

$\bar\theta(x+\lambda) - \bar\theta(x)$ independent of x.

$\Downarrow$

Partial derivatives of $\bar\theta$ are Λ_1-translation invariant.

$\Downarrow$

they define holomorphic functions on T_1, hence constant

$\Downarrow$

$\bar\theta$ is an affine map of the form $\bar\theta(x) = \tilde{\theta}(x) + y$

$\tilde\theta : V_1 \to V_2$ homo. & $y \in V_2$

$\Downarrow$

$\tilde\theta(\Lambda_1) \subset \Lambda_2$

$\Downarrow$

induces $\hat{\tilde\theta} : T_1 \longrightarrow T_2$ homo.

$$\begin{tikzcd} \hat{\tilde\theta}^* : H^0(\Omega^1_{T_2}) \arrow[r] \arrow[d, "\delta"', "\cong"] & H^0(\Omega^1_{T_1}) \arrow[d, "\cong", "\delta"'] \\ V_2^* \arrow[r, "\tilde{\bar\theta}^*"] & {} \end{tikzcd}$$

$\square$

THEOREM

X smooth projective variety

Called Albanese variety of X denoted by $Alb(X)$.

$\Rightarrow$ $\exists$ an abelian variety A & a morphism $a : X \to A$

Called Albanese morph.

with the following universal property :

for $\forall$ complex torus T & $\forall$ morph. $f : X \to T$

there exists a unique morph. $\tilde f : A \to T$ s.t.

$$\begin{tikzcd} X \arrow[r, "a"] \arrow[d, "f"'] & A \arrow[dl, "\exists! \tilde f"] \arrow[dl, phantom, bend right, "\circlearrowleft"] \\ T & \end{tikzcd}$$

Moreover, a induces an isom.

$a^* : H^0(\Omega^1_A) \xrightarrow{\sim} H^0(\Omega^1_X)$

$\underline{\underline{\text{pf}}}$. Step 1 : the image of $i : H_1(X, \mathbb{Z}) \longrightarrow H^0(\Omega^1_X)^*$

$\gamma \longmapsto i(\gamma)$

is a lattice in $H^0(\Omega^1_X)^*$ & $\left(\langle i(\gamma), \omega \rangle := \int_\gamma \omega \right)$

the quotient $H^0(\Omega^1_X)^* / i(H_1(X, \mathbb{Z}))$ is an abelian var

- Consider the period pairing

$$\langle -, - \rangle : H^0(\Omega^1_X) \times H_1(X, \mathbb{Z}) \longrightarrow \mathbb{C}$$
$$(\omega, \gamma) \longmapsto \int_\gamma \omega$$

this pairing is well-defined & $\mathbb{C}$-linear

$\omega = \sum_{k=1}^{n} \overset{\text{holo.}}{f_k(z)} dz^k$

$dz^k = dx^k + \sqrt{-1} dy^k$

$d\omega = \sum df_k \wedge dz^k$

f_k holo. $\Rightarrow$ Cauchy-Riemann eq $\Rightarrow$ $\bar\partial f_k = 0 \Rightarrow d\omega = 0$.

$\forall$ holomorphic 1-forms ω are closed ($d\omega = 0$) $\xRightarrow[\text{thm}]{\text{Stokes}}$ If $[\gamma_1] = [\gamma_2] \in H_1(X, \mathbb{Z})$ then $\exists$ 2-dim'l boudary submfd D s.t. $\partial D = \gamma_1 - \gamma_2$.

$\int_{\gamma_1} \omega - \int_{\gamma_2} \omega = \int_{\gamma_1 - \gamma_2} \omega = \int_{\partial D} \omega \overset{\text{Stokes}}{=} \int_D (d\omega) = 0$)

- $i(H_1(X,\mathbb{Z}))$ is a discrete subgroup of $H^0(\Omega_X^1)^*$

$\left.\begin{array}{l} H_1(X,\mathbb{Z}) \text{ finitely generated free abelian group of rank } 2g \\ H^0(\Omega_X^1)^* \ g\text{-dim'l } \mathbb{C}\text{-vector space, endowed with Euclidean topo.} \\ i: H_1(X,\mathbb{Z}) \longrightarrow H^0(\Omega_X^1)^* \text{ is continuous} \end{array}\right\} \Rightarrow$

the continuous image of $H_1(X,\mathbb{Z})$ f.g. abelian group. in $H^0(\Omega_X^1)^*$

$\Rightarrow$ $i(H_1(X,\mathbb{Z}))$ discrete $\Leftrightarrow$ it no torsions.

Claim: $i: H_1(X,\mathbb{Z}) \longrightarrow H^0(\Omega_X^1)^*$ injective. & hence $i(H_1(X,\mathbb{Z}))$ torsion-free

If $i(\gamma) = 0$ for some $\gamma \in H_1(X,\mathbb{Z})$.

For $\forall$ holo. 1-form $\omega \in H^0(\Omega_X^1)$, $\int_\gamma \omega = 0$

by Hodge decomposition

$$H^1(X,\mathbb{C}) = H^{1,0}(X) \oplus \overline{H^{1,0}(X)}$$

complex conjugate

$\Rightarrow$ for $\forall$ complex 1-form $\alpha \in H^1(X,\mathbb{C})$, $\int_\gamma \alpha = 0$.

By Poincaré duality, $\gamma = 0$.

($\langle\cdot,\cdot\rangle_{PD}: H^k(X,\mathbb{Z}) \times H_k(X,\mathbb{Z}) \to \mathbb{Z}$ is non-degenerate)

$(\alpha, \gamma) \longmapsto \int_\gamma \alpha$

- $\mathrm{Span}_{\mathbb{R}}\left(i(H_1(X,\mathbb{Z}))\right) = H^0(\Omega_X^1)^*$

$\left.\begin{array}{l} \dim_{\mathbb{R}} H^0(\Omega_X^1)^* = 2g \\ \operatorname{rank} i(H_1(X,\mathbb{Z})) = \operatorname{rank} H_1(X,\mathbb{Z}) = 2g \\ i(H_1(X,\mathbb{Z})) \underset{\text{discrete}}{\subset} H^0(\Omega_X^1)^* \end{array}\right\} \Rightarrow i(H_1(X,\mathbb{Z}))$ $\mathbb{R}$-spans $H^0(\Omega_X^1)$

(write $\Omega = H^0(\Omega_X^1)$ & $H := \underset{\text{lattice}}{\mathrm{Im}(i) \subset \Omega}$)

Step 2: define the morphism $\alpha: X \longrightarrow A := \Omega^*/H$

Fix a base point $x_0 \in X$. For $\forall$ point $x \in X$, let γ_x be a path joining x to x_0 in X & $a(\gamma_x) \in \Omega^*$ be the linear form $(\omega \longmapsto \int_{\gamma_x} \omega)$

If replacing γ_x by another path γ_x' joining x to x_0, we change $a(\gamma_x)$ by an element of H

$\Rightarrow$ the class $a(\gamma_x)$ in $A = \Omega^*/H$ depends only on x.

Call it $\alpha(x)$.

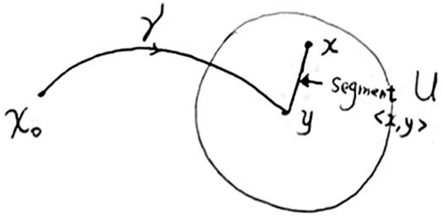
Step 3 : ∂ is analytic in a nbhd of a point $y \in X$.

Choose a path γ from x_0 to y & a nbhd U of y in X isomorphic to a ball B in $\mathbb{C}^n$ (will identify U with B)

$\partial : X \longrightarrow A$, $\quad x \longmapsto \partial(x)$

For $x \in U$, put $a(x) = a(\gamma_x)$

the path $\langle x, y \rangle \circ \gamma$ (segment)



$\rightsquigarrow a : U \longrightarrow \Omega^*$

$x \longmapsto a(\gamma_x)\left(\omega \mapsto \int_{\gamma_x} \omega\right)$

is an analytic morph.

$\Downarrow$

∂ analytic in U

& hence $\partial(x_0) = 0 \in A$ (base point)

$$\begin{tikzcd} X \arrow[r, "\partial"] & A \arrow[d, equal] \\ U \arrow[u, phantom, "\cup"] \arrow[r, "\partial|_U"] \arrow[dr, "a"'] & A = \Omega^*/H \\ & \Omega^* \arrow[u, "\pi"'] \end{tikzcd}$$

π quotient map

If we change base point x_0,

∂ is altered by a translation in A.

Step 4 : ∂ induces an isom. $\partial^* : H^0(\Omega^1_A) \longrightarrow H^0(\Omega^1_X)$

Since $\delta : \Omega \xrightarrow{\simeq} H^0(\Omega^1_A)$. $\rightsquigarrow$ suff. to show

$$\partial^*(\delta\omega) = \omega \text{ for } \forall \omega \in \Omega$$

locally on X, we can write $\partial = \pi \circ a : U \xrightarrow{a} \Omega^* \xrightarrow{\pi} A$ $\quad \left(A = \Omega^*/H\right)$

$$\partial^*(\delta\omega) = a^* \pi^*(\delta\omega) = a^*\big(d(\langle \omega, \cdot \rangle)\big) \in H^0(\Omega^1_U)$$

The value of this form at a point $x \in X$ is

$$d\big(\langle \omega, a(x) \rangle\big) = d\left(\int_{x_0}^{x} \omega\right) = \omega(x)$$

$$\Longrightarrow \quad \partial^*(\delta\omega) = \omega.$$

Step 5 : Universal property of $\partial : X \to A$

let $T = V/\Gamma$ be a complex torus & $f : X \to T$ a morph.

(Uniqueness of $\tilde{f}$)

$$\begin{tikzcd} X \arrow[r, "\partial"] \arrow[d, "f"'] & A \arrow[dl, "\tilde{f}"] \\ T & \end{tikzcd}$$

$\exists$ a commutative

$$\begin{tikzcd} H^0(\Omega^1_X) \arrow[dr, phantom, "\circlearrowleft"] & H^0(\Omega^1_A) \arrow[l, "\cong"', "\partial^*"] \\ H^0(\Omega^1_T) \arrow[u, "f^*"] \arrow[ur, "\tilde{f}^*"'] & \end{tikzcd}$$

$\Longrightarrow$ this determines $\tilde{f}^*$

by above lemma, $\tilde{f}$ determined up to translations.

since $f = \tilde{f} \circ \alpha$, $f(x_0) = \tilde{f}(\alpha(x_0))$ $\overset{\shortparallel}{\tilde{f}(0)}$

$\Rightarrow \tilde{f}$ unique.

(Existence of $\tilde{f}$) to construct $\tilde{f}: \underset{A}{\Omega^*/H} \xrightarrow{\Phi} \underset{T}{V/\Gamma}$

" $\Omega^* \longrightarrow V$ & $H \mapsto \Gamma$ "

again by above lemma, enough to show that the composition

$$u: V^* \xrightarrow{\delta} H^0(T, \Omega_T^1) \xrightarrow{f^*} \Omega$$

satisfies ${}^t u : \Omega^* \longrightarrow V$

$$\boxed{{}^t u(H) \subset \Gamma}$$

$i: H_1(X, \mathbb{Z}) \to \Omega^*$, $\gamma \mapsto i(\gamma) \in H \subset \Omega^*$

let $\gamma \in H_1(X, \mathbb{Z})$, $v^* \in V^*$, then

$$\langle {}^t u(i(\gamma)), v^* \rangle = \langle i(\gamma), u(v^*) \rangle$$
$$= \int_\gamma f^*(\delta v^*)$$
$$= \int_{f_*\gamma} \delta v^* \overset{*}{=} \langle h^{-1}(f_*\gamma), v^* \rangle$$

$\Rightarrow {}^t u(i(\gamma)) = h^{-1}(f_* \gamma) \in \Gamma$.

here, $h: \Gamma \longrightarrow H_1(T, \mathbb{Z})$ is the isom. given as follows ($T = V/\Gamma$ complex torus)

$\gamma \longmapsto h\gamma$

For $\forall \gamma \in \Gamma$, consider the path $t \longmapsto t\gamma$ $(0 \le t \le 1)$.

$$\int_{h\gamma} \delta x^* = \int_0^1 d\langle x^*, t\gamma \rangle \quad \text{for } \forall x^* \in V,\ \gamma \in \Gamma$$
$$= \langle x^*, \gamma \rangle.$$

$\delta: V^* \xrightarrow{\sim} H^0(\Omega_T^1)$

$\square$

Rmks

① $\dim \mathrm{Alb}(X) = \dim H^0(\Omega_X^1)$

In particular, if $H^0(\Omega_X^1) = 0$, then any morphism from X to a complex torus is trivial. (i.e. the image is a single point)

② if $f: X \to Y$ morph. of sm proj var.

$\rightsquigarrow \exists!\ \theta: \mathrm{Alb} X \longrightarrow \mathrm{Alb} Y$ s.t.

$$\begin{array}{ccc} X & \xrightarrow{\alpha_X} & \mathrm{Alb} X \\ f \downarrow & \circlearrowright & \downarrow \theta \\ Y & \xrightarrow{\alpha_Y} & \mathrm{Alb} Y \end{array}$$

← by the universal property of α_X.

③ By the universal property of Albanese morph.

$\mathrm{Alb}(X)$ is generated by $a(X)$, since the abelian subvar. of $\mathrm{Alb}(X)$ generated by $a(X)$ also satisfies the universal property.

In particular, if $\mathrm{Alb}(X) \neq \{0\}$ then $a(X)$ is not reduced to a point.

if $f: X \to Y$ surjective, then so is $\mathrm{Alb}X \xrightarrow{\mathrm{alb}_f} \mathrm{Alb}Y$

④ if X is a curve, then

$$\mathrm{Alb}X = JX.$$

⑤ $a_X : X \longrightarrow \mathrm{Alb}X$ (assume $q(X) = h^0(\Omega_X^1) > 0$)

define the Albanese dimension of X as

$$\dim a_X(X)$$

denoted by $\mathrm{alb.dim}(X)$ $\rightsquigarrow$ $\mathrm{alb.dim}X \in \{1, 2, \cdots, \dim X\}$

If $\mathrm{alb.dim}X = \dim X$, then X called of maximal Albanse dim. (mAd)

Prop S surface

$a: S \longrightarrow \mathrm{Alb}S$ the Albanese map

Suppose that $a(S)$ is a curve C

then C is a smooth curve of genus $q(S)$

& the fibres of a are connected.

(In this case, call $a: S \to C$ the Albanese fibration of S

lemma Suppose that the Albanese map a factorizes as

$$S \overset{f}{\twoheadrightarrow} T \xrightarrow{j} \mathrm{Alb}S$$

with f surjective.

then $\tilde{j}: \mathrm{Alb}(T) \longrightarrow \mathrm{Alb}S$ is an isom.

pf of lem.

$$\begin{tikzcd}
S \arrow[r, "a_S"] \arrow[d, "f"'] \arrow[dd, bend right=60, "a_S"'] & \mathrm{Alb}S \arrow[d, "a_f"] \arrow[ddl, bend left=60, "\text{must be id}"] \\
T \arrow[r, "a_T"] \arrow[d, "j"] & \mathrm{Alb}T \arrow[dl, "\tilde{j}"] \\
\mathrm{Alb}S &
\end{tikzcd}$$

□

<u>proof of Prop.</u>

$$S \xrightarrow[a_S]{} C \subset \mathrm{Alb}\,S$$

let N be the normalization of C

S normal $\Rightarrow$ a_S factorizes as $a_S : S \xrightarrow{f} N \xrightarrow{j} \mathrm{Alb}\,S$

with f surjective

$\Downarrow$ Lemma

$\tilde{j} : JN \xrightarrow{\cong} \mathrm{Alb}\,S$

$j : N \to \mathrm{Alb}\,S$ embedding $\Leftarrow \left\{ \begin{array}{l} \tilde{j} : JN \xrightarrow{\cong} \mathrm{Alb}\,S \\ a_N : N \hookrightarrow JN \text{ embedding} \end{array} \right.$

$\Downarrow$

$N = C$ i.e. C smooth curve of genus q

Consider the Stein factorization of a_S

$$a_S : S \xrightarrow{p} \tilde{C} \xrightarrow{g} C$$

replacing $\tilde{C}$ by its normalization if necessory.

WMA $\tilde{C}$ smooth.

Again by lemma,

$$\begin{tikzcd} S \arrow[r, "p"] & \tilde{C} \arrow[r, "g"] \arrow[d, hook] & C \arrow[d, hook] \\ & G : J\tilde{C} \arrow[r, "\sim"] & JC \end{tikzcd}$$ an isom.

$\Swarrow$

g an isom.

$\square$

(Application)

<u>Lemma</u> | S surface with $p_g = 0$, $q \geq 1$
$a : S \to \mathrm{Alb}\,S$ Albanese map
then $a(S)$ is a curve.

<u>Pf.</u> If $a(S)$ is a surface, then $a : S \to a(S)$ is generically finite & hence étale over an open subset $U \subseteq a(S)$

let $x \in U \subseteq a(S)$,

$a(S)$ is smooth at x

$\Downarrow$

Can take local coordinates $u_1, \dots, u_q$ for $\mathrm{Alb}\,S$ at x

s.t. $a(S)$ defined locally by $u_3 = \dots = u_q = 0$.

Since $\mathrm{Alb}\,S$ parallelizable $\Rightarrow \exists$ 2-form $\omega \in H^0(\Omega_A^2)$ s.t. ω & $du_1 \wedge du_2$ have the same value at x, but $a^*\omega$ globally 2-form on S & non-zero above x. $\lightning$

Invariants of Ruled Surfaces.

Fact For product schemes

(1) if X, Y schemes over a base scheme S, then

$$\Omega_{X\times_S Y/S} \xrightarrow{\simeq} p_1^*\Omega_{X/S} \oplus p_2^*\Omega_{Y/S}$$

(2) if X & Y nonsingular Var/k

then $\omega_{X\times Y} \cong p_1^*\omega_X \otimes p_2^*\omega_Y$.

$$H^0(X,\omega_X)\otimes H^0(Y,\omega_Y) \xrightarrow{\simeq} \underset{\substack{\text{\rotatebox{90}{$\cong$}} \\ H^0(X\times Y,\, \omega_{X\times Y})}}{H^0(p_1^*\omega_X \otimes p_2^*\omega_Y)}$$

Prop If S ruled surface over base curve C,

then $q(S) = g(C)$

$p_g(S) = 0$

$$\left.P_n(S) = 0 \quad \text{for } \forall n\geq 2\right\} \Rightarrow P_n(S)=0 \quad \forall n\geq 1$$

If S is geometrically ruled over C, then

$$K_S^2 = 8(1-g(C)), \quad b_2(S) = 0$$

Pf S ruled surface over C i.e. $S \overset{bir}{\sim} C\times\mathbb{P}^1$

$$H^0(\Omega_S^1) \cong H^0(\Omega^1_{C\times\mathbb{P}^1}) = H^0(\omega_C)\oplus H^0(\omega_{\mathbb{P}^1})$$

$$\Downarrow$$

$$q(S) = g(C)$$

$$H^0(\omega_S^{\otimes n}) \cong H^0(\omega_C^{\otimes n})\otimes H^0(\omega_{\mathbb{P}^1}^{\otimes n}) = 0$$

$$\Downarrow$$

$$P_n(S) = 0 \quad \forall n\geq 1$$

Recall that if $S \xrightarrow{\pi} C$ geometrically ruled, then

$S \cong \mathbb{P}_C(E)$ for some vector bundle E of rank 2 on C

$$0\to L\to \pi^*E \longrightarrow \mathcal{O}_S(1)\to 0$$

$$\Rightarrow L\otimes\mathcal{O}_S(1) \cong \Lambda^2(\pi^*E) \cong \pi^*(\Lambda^2E) \qquad [L] = -h+\pi^*e \in \mathrm{Pic}\,S$$

$h = [\mathcal{O}_S(1)] \in \mathrm{Pic}\,S$

$f = [\text{fibre}] \in \mathrm{Pic}\,S$

$e = [\Lambda^2E] \in \mathrm{Pic}\,C$

$$L.\mathcal{O}_S(1) = L^{-1}.\mathcal{O}_S(1)^{-1}$$

$$\overset{R.R.}{=} \chi(\mathcal{O}_S) - \chi(\pi^*E) + \chi(\pi^*(\Lambda^2E))$$

$$= 0$$

$$\Rightarrow h^2 = h.\pi^*e = \deg E \qquad f^2=0 \quad hf = 1 \qquad \mathrm{Pic}\,S = \pi^*\mathrm{Pic}\,C\oplus\mathbb{Z}h$$

$$H^2(S,\mathbb{Z}) = \mathbb{Z}h\oplus\mathbb{Z}f$$

$$\Rightarrow [K_S] = -2h + (\deg E + 2g(C)-2)f \in H^2(S,\mathbb{Z})$$

$$\Rightarrow K_S^2 = 4\deg E - 4(\deg E + 2g(C)-2) = 8(1-g(C)) \;\&\; b_2(S) = 2 \qquad \square$$

Numerical invariants of ~~ruled~~ surfaces

Numerical invariants

For any smooth projective surface S

$q(S) = h^1(S, \mathcal{O}_S)$ called irregularity of S

$p_g(S) = h^2(S, \mathcal{O}_S) \underset{\text{duality}}{\overset{\text{Serre}}{=\!=}} h^0(S, \omega_S)$ geometric genus

$P_n(S) = h^0(S, \omega_S^{\otimes n}) = h^0(S, nK_S)$ plurigenera of S
$(n \geq 1)$

$\chi(\mathcal{O}_S) = \sum (-1)^i h^i(\mathcal{O}_S) = 1 - q(S) + p_g(S)$ holomorphic Euler-Poincaré characteristic

$b_i(S) = \dim_{\mathbb{R}} H^i(S, \mathbb{R})$ i-th Betti number

$e(S) = \sum (-1)^i b_i(S)$ topological Euler-Poincaré characteristic

$$\left.\begin{array}{l} b_0 = b_4 = 1 \\ b_1 = b_3 \quad \text{(by Poincaré duality)} \end{array}\right\} \Rightarrow e(S) = 2 - 2b_1 + b_2$$

<u>Fact</u> $q(S) = h^0(S, \Omega_S^1) = \frac{1}{2} b_1(S)$
$b_2(S) = 2p_g(S) + h^{1,1}$ (by Hodge theory)

Noether's formula

$$\chi(\mathcal{O}_S) = \frac{1}{12}(e(S) + K_S^2)$$

<u>Prop</u> The integers $q(S)$, $p_g(S)$, $P_n(S)$ are birational invariants.

<u>Sketch proof</u>. On birational invariance of $q(S) = h^0(\Omega_S^1)$

let $\phi: S' \dashrightarrow S$ be a birational map, then it is a morphism $\phi: S' - \Sigma \longrightarrow S$ where $\Sigma \subset S'$ a finite set

For $\forall$ holomorphic 1-form $\omega \in H^0(\Omega_S^1)$,
the form $\phi^*\omega$ defines a rational 1-form on S' with poles lying in Σ.

Since the poles of a differential form are divisors,
$\phi^*\omega$ is in fact homorphic on the whole S'

$\Longrightarrow$ can define an injective map

$$\phi^*: H^0(\Omega_S^1) \longrightarrow H^0(\Omega_{S'}^1)$$

ϕ birational $\Rightarrow$ ϕ^* has an inverse $\Rightarrow$ $q(S) = q(S')$.

The birat'l invariance of p_g & P_n proved in the same way

$\square$

Minimal surfaces with $K_S^2<0$ are ruled.

lemma 1 If S surface with $p_g=0$, $q\geq 1$

then $K_S^2 \leq 0$

$K_S^2<0$ unless $q=1$ & $b_2=2$

Pf $b_1=2q$.

$$12\chi = K^2+e(S) = K^2+2-2b_1+b_2 = K^2+2-4q+b_2$$

$p_g=0 \longrightarrow \| \ 12-12q$

$$\Downarrow$$

$$K^2 = 10-8q-b_2$$

If $q\geq 2$, then $K_S^2<0$

If $q=1$, then $K_S^2 = 2-b_2$

Consider the Albanese morphism $a_S: S \longrightarrow E = \mathrm{Alb}\,S$

$\uparrow$ elliptic curve

$f\in H^2(S,\mathbb{Z})$ the class of a general fibre of a_S

h : the class of a hyperplane section.

$\left.\begin{array}{l} f^2=0 \\ hf>0 \end{array}\right\} \Rightarrow$ h & f are linearly independent in $H^2(S,\mathbb{Z})$

$\Rightarrow b_2\geq 2$

$K_S^2=0 \iff q=1$ & $b_2=2$ $\square$

lemma 2 S minimal surface with $K_S^2<0$

then $p_g=0$ & $q\geq 1$.

Pf • Suppose $p_g=h^0(K_S)\neq 0$. Say $D\in |K_S|$

$D=\sum n_i C_i>0$

$K_S^2 = K_S D = \sum n_i K_S C_i < 0 \Rightarrow K_S C_i < 0$ for some i.

$C_iC_j\geq 0$ $(i\neq j)$ $\qquad \| \ \sum n_j C_i C_j$

$$\Downarrow$$

$$C_i^2<0$$

by genus formula

$$2p_a(C_i)-2 = K_S C_i + C_i^2$$

$\Rightarrow p_a(C_i)=0$, $C_i^2=-1$

i.e. C_i (-1)-curve ↯ (S minimal)

Similarly, $P_n(S)=h^0(nK_S)=0$ for $\forall n\geq 1$.

• If $q(S)=0$, $P_2(S)=0$, then S rational by Castelnuovo's Rationality Criterion

$\Rightarrow K^2=8$ or 9 ↯ $\square$

Surfaces with $p_g=0$ and $q\geq 1$

<u>lemma</u>

(1) S surface with $p_g=0$, $q\geq 1$

then $K_S^2\leq 0$.

$K_S^2<0$ unless $q=1$ and $b_2=2$

(2) S minimal surface with $K_S^2<0$

then $p_g=0$ & $q\geq 1$.

<u>Prop</u> | let S be minimal surface with $K_S^2<0$
then S is ruled.

<u>Pf</u>. By above lemma, $p_g(S)=0$ and $q\geq 1 \Rightarrow a(S)$ is a curve B

$\Downarrow$

$a: S\to B$ is a fibration over a smooth genus q curve B

Assume that S not ruled.

Step 1: | let $C\subset S$ irreducible curve with $K_SC<0$ & $|K+C|=\emptyset$
then $a|_C$ is étale, and is an isom. if $q\geq 2$
(i.e. C is a section of a)
In particular, $g(C)=q$.

Indeed,

- Apply R-R. to K_S+C

$$0=h^0(K_S+C)\geq \underbrace{\chi(\mathcal{O}_S)}_{1-q}+\underbrace{\tfrac{1}{2}(C^2+K_SC)}_{p_a(C)-1} \Rightarrow \underline{p_a(C)\leq q}$$

- $\left.\begin{array}{l} S \text{ minimal} \\ C^2\geq 0 \end{array}\right\} \xRightarrow{\text{Zariski's lemma}}$ C cannot lie a reducible fibre of a

If C was a fibre of a, then $\left.\begin{array}{l} C^2=0 \\ \text{by assumption, } K_SC<0 \end{array}\right\} \Rightarrow \begin{array}{l} K_SC=-2 \\ p_a(C)=0 \end{array}$

i.e. C (-2)-curve

$\Downarrow$ Noether-Enriques

S ruled

$\Rightarrow C$ is a-horizontal, that is $a(C)=B$.

let $\nu: N\to C$ be the normalization of C

then a defines a ramified cover $N\to C\xrightarrow{a} B$ of deg d

by Riemann-Hurwitz formula,

$$2g(N)-2 = d\,(2g(B)-2)+r$$

r: # of branch points of $N\to B$

$\Rightarrow 1+d(q-1)\leq g(N)\leq p_a(C)\leq q \Rightarrow$ either $d=1$, $C=N$ or $q=1$, $C=N$

Step 2 : $\exists$ irreducible curve $C \subset S$ with $\begin{array}{l} |K_S + C| = \phi \\ K_S C < -1 \end{array}$

(pf of step 2)

Recall. lemma | S minimal surf with $K_S^2 < 0$
for $\forall\, m > 0$.
$\exists$ eff div. D on S s.t. $\begin{array}{l} K_S D \leq -m \\ |K_S + D| = \phi \end{array}$

$\Rightarrow$ $\exists$ divisor $D \geq 0$ s.t. $|K_S + D| = \phi$ & $K_S D < -1$

Say $D = \sum_{i=1}^{r} n_i C_i$ $(n_i > 0)$

removing some of the C_i if necessary.

WMA $KC_i < 0$ for all i.

<u>Claim</u> : D is then in fact irreducible.

(a) Suppose that $n_i \geq 2$ for some i. Then $|K_S + 2C_i| = \phi$

R.R. $\rightsquigarrow$

$$0 = h^0(2C_i + K_S) \geq \chi(2C_i + K_S)$$
$$\| $$
$$1 - q + 2C_i^2 + K_S C_i$$
$$\|$$
$$1 - q + 2(C_i^2 + K_S C_i) - KC_i \; \lightning$$

by Step 1. $C_i^2 + K_S C_i = 2(q-1) \longrightarrow \| \; 3(q-1) - KC_i$

(b) Suppose that $r \geq 2$. Then $|K_S + C_1 + C_2| = \phi$

by Step 1 again.

$$0 = h^0(K_S + C_1 + C_2) = h^1(K_S + C_1 + C_2) + 1 - q + \tfrac{1}{2}(C_1^2 + KC_1)$$
$$+ \tfrac{1}{2}(C_2^2 + KC_2) + C_1 C_2$$
$$= (q-1) + h^1(K_S + C_1 + C_2) + C_1 C_2$$

which is impossible unless $C_1 C_2 = h^1(K_S + C_1 + C_2) = 0$

But if $C_1 \cap C_2 = \phi$, then Consider standard exact seq.

$$0 \to \mathcal{O}_S(-C_1 - C_2) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2} \to 0$$

$0 \neq t_1 \in H^0(\mathcal{O}_S(C_1))$

$0 \neq t_2 \in H^0(\mathcal{O}_S(C_2))$

taking cohomology.

$$0 \to \overset{0}{H^0(\mathcal{O}_S(-C_1 - C_2))} \to \overset{1}{H^0(\mathcal{O}_S)} \to \overset{2}{H^0(\mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2})} \to H^1(\mathcal{O}_S(-C_1 - C_2))$$
$$\downarrow$$
$$H^1(\mathcal{O}_S)$$

$\Rightarrow$ $H^1(\mathcal{O}_S(-C_1 - C_2)) \neq 0$

Serre duality $\Rightarrow$ $h^1(K_S + C_1 + C_2) \neq 0$ $\lightning$.

Step 3 : We get a contradiction

let C be an irred curve on S with $K_S C < -1$ & $|K_S + C| = \phi$

(Case 1) suppose C is a section of $a : S \to B$

$$h^0(C) \overset{\text{R.-R.}}{\geq} 1 - q + \frac{1}{2}(C^2 - K_S C) \overset{p_a(C)=q \Rightarrow C^2 + KC = 2(q-1)}{=} -K_S C \geq 2$$

in other words, $C \in |C|$ moves.

let F be a generic fibre of a, then

the point $C \cap F$ moves linearly on F

$\Downarrow$

F must be rational $\lightning$

(Case 2) $q = 1$ & $a|_C$ is étale.

$$\begin{tikzcd} C \arrow[dr, "e"] \arrow[drr, bend left, "i\ (\text{inclusion})"] \arrow[ddr, bend right, "id_C"'] & & \\ & S \times_B C \arrow[r] \arrow[d] & S \arrow[d, "a"] \\ & C \arrow[r, "a|_C"'] & B \end{tikzcd}$$

$e : C \longrightarrow S \times_B C$ is a section of $S \times_B C \to C$

let S' denote the connected component of $S \times_B C$ that contains $e(C) := C'$, say.

then the projection $\pi : S' \longrightarrow S$ is étale

$$\begin{tikzcd} S \times_B C & & \\ S' \arrow[u, phantom, "\cup"] \arrow[r, "\pi"] \arrow[d] & S \arrow[d, "a"] \\ C \arrow[u, bend left, "e"] \arrow[ur, "i"] \arrow[r, "a|_C"'] & B \end{tikzcd}$$

$\Rightarrow \Omega^1_{S'} \cong \pi^* \Omega^1_S$ & $K_{S'} \sim \pi^* K_S$

$\Rightarrow K_{S'} C' = \deg_{C'}(e^* K_{S'}) = \deg_C(i^* K_S) = K_S C < -1$

Clearly $p_g(S') = 0$.

R.-R. $\rightsquigarrow$ $h^0(C') \geq \chi(\mathcal{O}_{S'}) - 1 + g(C') - K_{S'} C'$

$\chi(\mathcal{O}_{S'}) = (\deg \pi)\, \chi(\mathcal{O}_S) = 0$

$\Rightarrow h^0(C') \geq 2$

$\lightning$

To complete the classification of surfaces with $p_g = 0$ & $q \geq 1$.

We must consider the case $\left| \; K_S^2 = 0,\ q = 1,\ b_2 = 2. \right.$

Prop S minimal surf with $p_g=0$, $q=1$ & $K_S^2=0$

$a: S \longrightarrow B$ the Albanese map

g : genus of a generic fibre of a

then if $g \geq 2$, $\Rightarrow$ a is smooth

if $g=1$ $\Rightarrow$ singular fibres of a are of the form $F_b = mE$, where E smooth elliptic curve.

pf We know that $b_2 = 2$.

(Step 1) Claim: a has irreducible fibres.

Suppose that some fibre contains two irred. components F_1, F_2

let H be a hyperplane section of S, and F a generic fibre of a.

Suff to show F_1, F_2, H are linearly independent in $H^2(S,\mathbb{Z})$

Suppose $aF_1 + bF_2 + cH = 0 \in H^2(S,\mathbb{Z})$

- If $c \neq 0$, then $F \cdot (aF_1 + bF_2 + cH) = 0 \in \mathbb{Z}$

$$\begin{matrix} F \cdot (aF_1 + bF_2 + cH) = 0 \in \mathbb{Z} \\ \| \\ cHF \quad \lightning \text{ (since } H \text{ ample)} \end{matrix}$$

So $\underline{\underline{c=0}}$. $\Rightarrow aF_1 + bF_2 = 0 \in H^2(S,\mathbb{Z})$

$\Rightarrow F_2 = rF_1$ for some $r \in \mathbb{Q}$

$$\left.\begin{matrix} \Rightarrow HF_2 = rHF_1 \\ H \text{ ample} \end{matrix}\right\} \Rightarrow r > 0$$

$$\left.\begin{matrix} F_1F_2 = rF_1^2 \\ F_1^2 < 0 \text{ (Zariski lemma)} \\ F_1F_2 \geq 0 \end{matrix}\right\} \Rightarrow r \leq 0$$

$\lightning$

(Step 2) multiple fibres of a

let $F_s = mC$ for some irreducible curve C, $m \geq 1$.

then $\boxed{e(F_s)} = e(C) \geq 2\chi(\mathcal{O}_C)$

$$p_a(C) = 1 + \tfrac{1}{2}(C^2 + K_S C) = (-1)^1(\chi(\mathcal{O}_C) - 1) = 1 - \chi(\mathcal{O}_C)$$

$$2\chi(\mathcal{O}_C) = -C^2 - K_S C = -K_S C = -\frac{1}{m} F_s K_S = -\frac{1}{m} F_\eta K_S = \frac{2}{m}\chi(\mathcal{O}_{F_\eta}) = \boxed{\frac{1}{m} e(F_\eta)}$$

$g \geq 1 \Leftrightarrow e(F_\eta) = 2\chi(\mathcal{O}_{F_\eta}) = 2 - 2g \leq 0$

$\Downarrow$

$e(F_s) \geq e(F_\eta)$

and "=" holds $\Leftrightarrow \begin{cases} 2\chi(\mathcal{O}_C) = e(C) \overset{\text{i.e.}}{\longleftrightarrow} C \text{ smooth} \\ \frac{1}{m} e(F_\eta) = e(F_\eta) \leftrightarrow \text{either } m=1 \text{ or } g=1 \end{cases}$

(Step 3) Topology of S restricts to the case $e(F_s) = e(F_\eta)$

let $\Sigma := \{ s \in B \mid \text{fibre } F_s \text{ over } s \text{ is singular} \}$ & $s \in \Sigma$, we have

$$\begin{cases} e(F_s) - e(F_\eta) \geq 0 \\ \text{"="} \text{ holds} \Leftrightarrow F_s = mE, \ E \text{ smooth elliptic curve } \& \ \underset{g(F_\eta)}{\overset{\|}{g}} = 1. \end{cases}$$

Now

$$\underset{\underset{\underset{0}{\|}}{2-2b_1+b_2}}{\overset{}{\underset{\|}{e(S)}}} = \underset{\underset{0}{\|}}{e(B)} e(F_\eta) + \sum_{s\in\Sigma} \left(e(F_s) - e(F_\eta) \right)$$

$$\Rightarrow \sum_{s\in\Sigma} \left(e(F_s) - e(F_\eta) \right) = 0$$

$\Rightarrow$ if $g=1$, every singular fibre of the form $F_b = mE$, E smooth elliptic

if $g \geq 2$, the Albanese fibration a is smooth.

$\square$

If a fibration admits multiple fibres, we can reduce to a smooth fibration:

<u>Lemma</u> $f: S \to B$ a morphism from surface S onto a smooth curve B whose fibres are either smooth or multiples of smooth curves

$\Rightarrow$ $\exists$ ramified Galois cover $\pi: B' \to B$ with Galois group G, say, a surface S' & a commutative diagram

$$\begin{tikzcd} S' \arrow[r, "\pi'"] \arrow[d, "f'"'] \arrow[dr, phantom, "\circlearrowright"] & S \arrow[d, "f"] \\ B' \arrow[r, "\pi"'] & B \end{tikzcd}$$

Such that the action of G on B' lifts to S', π' induces an isom. $S'/G \xrightarrow{\sim} S$, & $f': S' \to B'$ is smooth.

<u>Pf</u>. Enough to elimate each multiple fibre by taking successive branched covers, So the lemma reduces to the following local version:

lemma $\{z \in \mathbb{C} \mid |z| < 1\} = \Delta \subset \mathbb{C}$ unit disc

U non-compact smooth analytic surface

$f : U \longrightarrow \Delta$ a morph. that is smooth over $\Delta - \{0\}$

s.t. $f^* 0 = m\, C$ for some smooth curve $C \subset U$

let $\partial : \Delta \longrightarrow \Delta$ morphism

$z \longmapsto z^m$

$\widetilde{U} := U \times_\Delta \Delta$

U' the normalization of $\widetilde{U}$

$$\begin{tikzcd}
U' \arrow[r, "\text{normal}"] \arrow[rr, bend left, "\partial'"] \arrow[dr, bend right, "f'"'] & \widetilde{U} \arrow[r, "\widetilde{\partial}"] \arrow[d] \arrow[dr, phantom, "\square"] & U \arrow[d, "f"] \\
& \Delta \arrow[r, "\partial"] & \Delta
\end{tikzcd}$$

the group μ_m of m-th roots of unity $\curvearrowright \Delta$

$(\zeta, z) \longmapsto \zeta z$

and so $\mu_m \curvearrowright \widetilde{U}$ via the 2nd factor.

and so $\mu_m \curvearrowright U'$. ∂' induces an isom. $U'/\mu_m \xrightarrow{\sim} U$

The fibration $f' : U' \longrightarrow \Delta$ is smooth.

Pf. Since the questions are local on U, WMA $\exists$ local coordinates x, y on U s.t.

$$\begin{tikzcd}
U \arrow[d, "f"] \\
\Delta
\end{tikzcd} \qquad f(x,y) = x^m$$

& C is defined by $x = 0$.

$$\begin{tikzcd}
\widetilde{U} \arrow[r] \arrow[d] & U \arrow[d, "f"] & (x,y) \arrow[d, mapsto] \\
\Delta \arrow[r, "\partial"] & \Delta & x^m \\
z \arrow[r, mapsto] & z^m &
\end{tikzcd}$$

$\widetilde{U} \cong U \times_\Delta \Delta$

then $\widetilde{U} = \{(x,y,z) \in U \times \Delta \mid x^m = z^m\}$

$= \{(x, y, \zeta x) \in U \times \Delta \mid (x,y) \in U,\ \zeta \in \mu_m\}$

$= \bigcup_{\zeta \in \mu_m} U_\zeta$

via $\widetilde{\partial} : \widetilde{U} \longrightarrow U$, each U_ζ isomorphic to U

$\widetilde{U}$ is the union of the m var. U_ζ identified along the line $x = 0$.

$U' \xrightarrow{\text{normal}} \widetilde{U}$

U' is the disjoint union of the U_ζ

$\mu_m \curvearrowright U'$ by interchanging the components

Identifying U_ζ with $U \rightsquigarrow f' : U_\zeta \longrightarrow \Delta$

$$(x,y) \longmapsto \zeta x$$
$$\begin{matrix} \mathrel{\cong} \\ (x,y,\zeta x) \end{matrix} \quad \uparrow \text{ smooth}$$

$$(x,y,\zeta x) \longmapsto (x,y)$$

$$\begin{tikzcd}
U' \arrow[r] & \tilde{U} \arrow[r] \arrow[d] \arrow[d, bend right] & U \arrow[d, "f"] \arrow[d, bend left] \\
& \Delta \arrow[r] & \Delta
\end{tikzcd}$$

$$\zeta x \longmapsto x^m = (\zeta x)^m$$

□

Prop. $f : S \longrightarrow B$ smooth morphism

F a fibre of f

Assume either $\begin{cases} g(B) = 1 \\ g(F) \geq 1 \end{cases}$, or $\{ g(F) = 1$.

then $\exists$ étale cover $B' \longrightarrow B$ s.t. the fibration

$f' : \underset{S \times_B B'}{\overset{\parallel}{S'}} \longrightarrow B'$ is trivial, i.e. $S' \cong B' \times F$

Furthermore, can take the cover $B' \to B$ to be Galois with the group G s.t. $S \cong (B' \times F)/G$.

Pf. Set-up : T variety

a curve of genus g over T is a smooth morphism $f : X \to T$ whose fibres are curves of genus g

(Recall Ehresmann's fibration theorem

$f : M \to N$ smooth proper submmersion between smooth mfds (tangent map surj at each point)

N connected

$\Rightarrow$ f is a locally C^∞-fibre bundle.)

$\Rightarrow$ In particular f is a topological fibre bundle

$\Downarrow$

for each $t \in T$, $\exists$ open nbhd $U \ni t$ s.t. $f^{-1}(U) \overset{\text{homeo.}}{\sim} U \times F$

$$R^1 f_* (\mathbb{Z}/n\mathbb{Z})(U) \cong H^1(f^{-1}(U), \mathbb{Z}/n\mathbb{Z})$$
$$\cong H^1(U \times F, \mathbb{Z}/n\mathbb{Z})$$
$$\overset{\text{Leray-Hirsch}}{\cong} \underbrace{H^1(U, \mathbb{Z}/n\mathbb{Z})}_{=0 \ \leftarrow\ U \text{ contractible}} \otimes H^0(F, \mathbb{Z}/n\mathbb{Z}) \oplus H^0(U, \mathbb{Z}/n\mathbb{Z}) \otimes H^1(F, \mathbb{Z}/n\mathbb{Z})$$
$$\cong H^0(U, \underbrace{H^1(F, \mathbb{Z}/n\mathbb{Z})}_{\text{local constant sheaf}})$$

$R^1 f_* (\mathbb{Z}/n\mathbb{Z})$ locally constant $\Longleftarrow$

for all n.

(cup product)

$\Rightarrow \exists$ a symplectic form ~~on~~ $R^1f_*(\mathbb{Z}/n\mathbb{Z}) \otimes R^1f_* \mathbb{Z}/n\mathbb{Z} \xrightarrow{\cup} (\mathbb{Z}/n\mathbb{Z})_T$ on $R^1f_* \mathbb{Z}/n\mathbb{Z}$

Constant sheaf

$\Updownarrow$

i.e. at each point $t \in T$

$$\begin{tikzcd} X \arrow[d, "f"'] \\ T \end{tikzcd}$$

- $H^1(X_t, \mathbb{Z}/n\mathbb{Z})$ & symplectic form on it
- $\rho: \pi_1(T,t) \longrightarrow \mathrm{Aut}(H^1(X_t, \mathbb{Z}/n\mathbb{Z}))$ preserving the symplectic form on $H^1(X_t, \mathbb{Z}/n\mathbb{Z})$

C (locally) connected,

locally constant sheaf $\mathcal{F}$ on C $\Leftrightarrow$ $\pi_1(C, x_0)$-rep.

i.e. $\exists$ group homo $\rho: \pi_1(C, x_0) \to GL(\mathcal{F}_{x_0})$

Symplectic form on a ~~locally constant~~ sheaf $\mathcal{F}$ is a sheaf isom.

$\omega: \mathcal{F} \otimes_{\underline{k}} \mathcal{F} \longrightarrow \underline{k}$ (locally constant)

satisfying
- bi-linearity
- anti-symmetry
- non-degenerate
- compactible $\rho(\gamma) \in Sp(\mathcal{F}_{x_0}, \omega_{x_0})$

Now endow the constant sheaf $(\mathbb{Z}/n\mathbb{Z})^{2g}_T$ with this standard symplectic form.

Recall a J_n-rigidified curve of genus g over T is a curve of genus g over T $(f: X \to T)$ together with a symplectic ~~$\emptyset$~~ isom.

$$(\mathbb{Z}/n\mathbb{Z})^{2g}_T \xrightarrow{\sim} R^1f_* \mathbb{Z}/n\mathbb{Z}$$

a curve of genus g over T can be J_n-rigidified if $\pi_1(T,t) \curvearrowright H^1(X_t, \mathbb{Z}/n\mathbb{Z})$ trivially.

Since $\mathrm{Aut}(H^1(X_t, \mathbb{Z}/n\mathbb{Z}))$ finite

$\rho: \pi_1(T,t) \longrightarrow \mathrm{Aut}(H^1(X_t, \mathbb{Z}/n\mathbb{Z}))$

$\Rightarrow$ $\ker \rho$ has finite index in $\pi_1(T,t)$

$\Downarrow$

$\exists$ étale cover $T' \longrightarrow T$ s.t.

$$\begin{tikzcd} X' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\square"] & X \arrow[d, "f"] \\ T' \arrow[r] & T \end{tikzcd}$$

$f': X' \to T'$ is J_n-rigidifiable $/T'$

Now let $n \geq 3$.

J_n-rigidification eliminates automorphism $\rightsquigarrow$ $\exists$ universal J_n-rigidified curve of genus g

$$U_{g,n} \longrightarrow T_{g,n}$$

i.e. $\forall$ J_n-rigidified curve of genus g $f: X \to T$ over T is the pullback

$$\begin{tikzcd} X \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & U_{g,n} \arrow[d] \\ T \arrow[r, "\exists !"'] & T_{g,n} \end{tikzcd}$$

---

the spaces $T_{g,n}$ are quasi-projective var.

~~we~~ We shall need the following properties:

① For $g \geq 2$, no non-constant analytic morph. $h: C \longrightarrow T_{g,n}$

② For $g = 1$, no non-constant analytic morph. $X \longrightarrow T_{1,n}$ (compact, connected var)

For ①.

$$\begin{tikzcd} & T_g \arrow[d] \\ \mathbb{C} \arrow[r, "h"'] & T_{g,n} \end{tikzcd}$$

Teichmüller space (universal cover of $T_{g,n}$)

bounded domain

$\Rightarrow$ no non-trivial morph. $\mathbb{C} \to T_g$.

One can also consider the space

$A_{g,n}$ (classified J_n-rigidified ppav of dim g)

& its universal cover H_g (bounded domain)

$$\begin{tikzcd} H_g \arrow[d] \\ A_{g,n} \end{tikzcd}$$

Applying Torelli theorem $\Rightarrow$ $T_{g,n} \longrightarrow A_{g,n}$ finite

$$t \longmapsto J(U_t)$$

$$\begin{tikzcd} & & H_g \arrow[d] \\ \mathbb{C} \arrow[r] & T_{g,n} \arrow[r] & A_{g,n} \end{tikzcd}$$

For ②

j-invariant defines a holomorphic function on X

$\rightsquigarrow$ it must be constant.

back to the proof of this prop.

$f: S \longrightarrow B$ smooth morphism of genus g

for $n \geq 3$. $\exists$ étale cover $B' \longrightarrow B$ s.t.

$$\begin{tikzcd} S' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & S \arrow[d, "f"] \\ B' \arrow[r, "\text{étale}"'] & B \end{tikzcd}$$

the curve $S' \to B'$ is J_n-rigidifiable

Choose some J_n-rigidification, we get a morphism

$h: B' \longrightarrow T_{g,n}$ s.t.

$$\begin{tikzcd} S' \arrow[r] \arrow[d, "f'"'] \arrow[dr, phantom, "\lrcorner"] & U_{g,n} \arrow[d] \\ B' \arrow[r, "h"'] & T_{g,n} \end{tikzcd}$$

If $g(B)=1$, then $g(B')=1$

If $g \geq 2$, then by property ①,

$\Downarrow$

h trivial

universal cover

$$\begin{tikzcd} \mathbb{C} \arrow[d] & \\ B' \arrow[r, "h"'] & T_{g,n} \end{tikzcd}$$

If $g=1$, then h is trivial by property ②. $\square$

Cor. S minimal non-ruled surf with $p_g=0$, $q=1$, $K_S^2=0$

$\Rightarrow$ $\exists$ two curves B, F of genus ≥ 1

$\exists$ finite group $G \subset \mathrm{Aut}(B)$ acting on $B\times F$ with

$$g(b,f) = (gb, f)$$

such that
- $S \cong (B\times F)/G$
- B/G elliptic
- if $g(F)\geq 2$, then B elliptic & G a group of translations of B.

lemma. B, F curves of genus $g \geq 1$

$G \subset \mathrm{Aut}(B)$

$G \curvearrowright B\times F$ compatibly with its action on B.

Then

(1) If $g(F)\geq 2$, then $G \curvearrowright F$

& $g.(b,f) = (gb, gf)$

(2) If $g(F)=1$, then $\exists$ étale cover $\tilde{B}\to B$ & a group H acting on $\tilde{B}$ & F s.t. $\tilde{B}/H \cong B/G$

$(\tilde{B}\times F)/H \cong (B\times F)/G$.

Pf. Step 1: For $g\in G$, $b\in B$, $f\in F$

$$g.(b,f) = (gb, \phi_g(b).f)$$

where $\phi_g(b)\in \mathrm{Aut}(F)$ depending continuously on B.

If $g(F)\geq 2$, $\mathrm{Aut}(F)$ finite $\Rightarrow$ $\phi_g(b)$ independent of b

$\Rightarrow$ (1) holds.

So suppose that F elliptic. Fix an origin $o\in F$.

then $\phi_g(b)$ is of the form

$$\begin{aligned}\phi_g(b) : F &\longrightarrow F\\ f &\longmapsto a_g(b).f + t_g(b)\end{aligned}$$

where $a_g(b) \in \mathrm{Aut}_{AV}(F)$ & $t_g(b)$ is a translation

finite group

$\Longrightarrow$ $a_g(b) = a_g$ is independent of b.

& $t_g : B \longrightarrow F$ is a morphism
$b \longmapsto t_g(b)$

For $g, h \in G$, we have

$$a_{gh} = a_g a_h \Rightarrow \underbrace{a : G \longrightarrow \mathrm{Aut}_{AV}(F)}_{\text{group homo.}}$$

Step 2 : $\exists$ a morphism $\rho : B \longrightarrow F$ & an integer n s.t.

$$\rho(gb) - a_g . \rho(b) = n\, t_g(b) \quad \text{for } \begin{cases} b \in B \\ g \in G. \end{cases}$$

<u>Pf</u>. Recall $\exists$ canonical isom. $F \longrightarrow \mathrm{Pic}^0(F)$

$$f \longmapsto [f] - [0]$$

Then $[f_1] + \cdots + [f_r] \underset{\text{lin}}{\sim} (r-1)[0] + \left[\Sigma f_i\right]$ for $f_i \in F$

let $\theta \in \mathrm{Aut}(F)$ given by $\theta(f) = a(f) + t$, $\begin{cases} a \in \mathrm{Aut}_{AV}(F) \\ t \in F \end{cases}$

If $D = (n-1)[0] + f$, then

$$\theta^* D = (n-1)[\theta^{-1}(0)] + [\theta^{-1}(f)]$$
$$= (n-1)[-a^{-1}(t)] + [a^{-1}(f) - a^{-1}(t)]$$
$$\underset{\text{lin.}}{\sim} (n-1)[0] + [a^{-1}(f) - n a^{-1}(t)]$$

let H be a hyperplane section of $B \times F$, the bundle

$$L := \mathcal{O}_{B\times F}\left(\sum_{g \in G} g^* H\right) \text{ is } G\text{-inv.}$$

For $b \in B$,

put $\underset{\substack{\uparrow \\ \text{a line bundle on } F \text{ of deg } n > 0.}}{L_b} := L \otimes \mathcal{O}_{\{b\}\times F} = L|_{\{b\}\times F}$

L G-inv. $\Rightarrow L_b = (g^* L) \otimes \mathcal{O}_{\{b\}\times F} = \phi_g(b)^* L_{gb}$

define $\rho : B \rightarrow F$ by

$$L_b = \mathcal{O}_F\left((n-1)[0] + [\rho(b)]\right)$$

$$\rightsquigarrow \rho(b) = a_g^{-1} \cdot \rho(gb) - n\, a_g^{-1} . t_g(b)$$

$\square$

Step 3

(baby case) $n = 1$.

define $u \in \mathrm{Aut}(B \times F)$ by

$$u . (b, f) = (b, f - \rho(b))$$

by step (2), $u g u^{-1}(b, f) = (gb, a_g . f)$

i.e. u defines an isom. $(B\times F)/G \xrightarrow{\sim} (B\times F)/H$

where $H = u G u^{-1} \curvearrowright B \times F$ diagonally.

$\square$

(general case)

Consider the (possibly disconnected) étale cover

$$\pi : \widetilde{B} \longrightarrow B$$

induced by

$$\begin{tikzcd} \widetilde{B} \arrow[r, "\pi"] \arrow[d, "\widetilde{\xi}"'] \arrow[dr, phantom, "\lrcorner"] & B \arrow[d, "\xi"] \\ F \arrow[r, "n"'] & F \end{tikzcd}$$

$\Rightarrow$ $\left.\begin{array}{l} \widetilde{B} \subset B \times F \\ \text{step (2)} \end{array}\right\} \Rightarrow \widetilde{B}$ G-stable.

- the group F_n (of torsion n points in F) $\curvearrowright \widetilde{B}$ by

$$\varepsilon.(b, f) = (b, f + \varepsilon)$$

Set

$$H = \langle G, F_n \rangle \subset \mathrm{Aut}(\widetilde{B})$$

then

$$g \varepsilon g^{-1} = a_g . \varepsilon \quad \text{for } \begin{array}{l} g \in G \\ \varepsilon \in F_n \end{array}$$

$\Rightarrow$ split exact sequence

$$1 \longrightarrow F_n \longrightarrow H \xrightarrow{v} G \longrightarrow 1$$

Make H acting on $\widetilde{B} \times F$ by

$$h(\tilde{b}, f) = (h\tilde{b}, \phi_{vh}(\pi\tilde{b}).f)$$

$$\widetilde{B}/F_n \cong B \Rightarrow \widetilde{B}/H \cong B/G$$

$$(\widetilde{B} \times F)/H \cong (B \times F)/G$$

Consider $u \in \mathrm{Aut}(\widetilde{B} \times F)$ defined by

$$u(\tilde{b}, f) = (\tilde{b}. f - \widetilde{\xi}(\tilde{b}))$$

then

$$u h u^{-1}(\tilde{b}, f) = (h\tilde{b}, a_{vh} \cdot f + \underbrace{\theta_h(\tilde{b})}_{a_{vh} \cdot \widetilde{\xi}(\tilde{b}) - \widetilde{\xi}(h\tilde{b}) + t_{vh}(\pi\tilde{b})})$$

Now by step (2) again, $n.\theta_h(\tilde{b}) = 0$.

$\Downarrow$ (i.e. $\theta_h(\tilde{b}) \in F_n$)

θ_h independent of $\tilde{b}$

the action of $H \curvearrowright \widetilde{B} \times F$, after shifting by u, is of the required form.

- If $\widetilde{B}$ not connected, take a connected component $\widetilde{B}_0$, & H_0 the subgroup of H preserving B_0, then $(\widetilde{B}_0 \times F)/H_0 \cong (\widetilde{B} \times F)/H$. $\square$

Theorem S minimal non-ruled surface with $p_g = 0$, $q \geq 1$

then $S \cong (B \times F)/G$, where

$$\left(\begin{array}{l} \begin{cases} B, F \text{ smooth irrational curves} \\ G \text{ finite group acting faithfully on } B \,\&\, F \end{cases} \\ \qquad \text{s.t. } B/G \text{ elliptic} \\ \qquad\quad\;\; F/G \text{ rational } \& \\ \text{one of the following conditions holds:} \\ \quad (1)\ B \text{ elliptic } \& \ G \text{ a group of translations of } B \\ \quad (2)\ F \text{ elliptic } \& \ G \curvearrowright B \times F \text{ freely.} \end{array}\right)$$

Conversely, every surfaces with these properties is minimal, non-ruled & $p_g = 0$, $q = 1$, $K^2 = 0$.

<u>Pf</u>. let S minimal non-ruled surface with $p_g = 0$, $q \geq 1$.

then $K_S^2 = 0$ & $q = 1$.

By previous lemmas, $S \cong (B \times F)/G$ $\begin{cases} G \curvearrowright B \,\&\, \curvearrowright F \\ B/G \text{ elliptic} \end{cases}$

Moreover, either B elliptic

or F elliptic

In each cases, $G \curvearrowright B \times F$ freely $\Rightarrow \pi : B \times F \to S$ smooth, étale

So suffices to detect when $(B \times F)/G$ has $p_g = 0$, $q = 1$.

- If $(B \times F)/G \cong S$ contains a rational curve C, then $\pi^{-1}(C)$ is a union of rational curves, each of which would maps surjectively to either B or F $\lightning$.

  i.e. it is minimal & non-ruled.

- Set $\tilde{S} = B \times F$

$$H^0(\Omega^1_{\tilde{S}}) \cong H^0(\omega_B) \oplus H^0(\omega_F) \Rightarrow q(\tilde{S}) = g(B) + g(F)$$

$$H^0(\Omega^2_{\tilde{S}}) \cong H^0(\omega_B) \otimes H^0(\omega_F) \Rightarrow p_g(\tilde{S}) = g(B)g(F)$$

$$\rightsquigarrow \chi(\mathcal{O}_{\tilde{S}}) = \chi(\mathcal{O}_B)\chi(\mathcal{O}_F) = 0$$

- If B (resp. F) elliptic, then $\Omega^2_{\tilde{S}} \cong q^*\omega_F$ (resp. $p^*\omega_B$) $\Rightarrow K_{\tilde{S}}^2 = 0$.

  $\Rightarrow \chi(\mathcal{O}_S) = 0$, $K_S^2 = 0$ since $\tilde{S} \to S$ étale.

$$H^0(\Omega^1_S) \overset{\pi\ \text{étale}}{\cong} H^0(\Omega^1_{\tilde{S}})^G$$
$$\cong H^0(\omega_B)^G \oplus H^0(\omega_F)^G$$
$$\cong \left(H^0(\omega_{B/G}) \oplus H^0(\omega_{F/G}) \right)$$

B/G elliptic $\Rightarrow$ $q(S) = 1 \Leftrightarrow F/G$ rational

in this case, $p_g(S) = 0$. (since $\chi(\mathcal{O}_S) = 0$)

$\blacksquare$

Calculate the plurigenera P_n of surfaces $(B\times F)/G$

Goal $\rightsquigarrow$ distinguish them numerically from elliptic ruled surfaces

Prop let $S=(B\times F)/G$ be a minimal non-ruled surface with $p_g=0$, $q\geq 1$.

then

(1) $P_4\neq 0$ or $P_6\neq 0$. In particular $P_{12}\neq 0$

(2) If B or F not elliptic, then

$\exists$ infinite increasing sequence $\{n_i\}$ of integers s.t.

the sequence $\{P_{n_i}\}\to+\infty$

(3) If B and F are elliptic, then $4K_S\underset{\text{lin}}{\sim}0$ or $6K_S\underset{\text{lin}}{\sim}0$. In particular $12K_S\underset{\text{lin}}{\sim}0$.

Pf. "(1) $\Rightarrow$ (3)" if B & F are elliptic, then $K_{B\times F}$ trivial.

If $D\in|4K_S|$, then $\pi^*D\sim 0\Rightarrow D\sim 0$
(resp. $|6K_S|$)

Case I (B elliptic) $\qquad \tilde{S}=B\times F\xrightarrow{\pi}S=(B\times F)/G$

One has

$$H^0(S,\Omega_S^2)^{\otimes k}\cong\left(H^0(\tilde{S},\Omega_{\tilde{S}}^2)^{\otimes k}\right)^G$$
$$\cong\left[H^0(B,\omega_B^{\otimes k})\otimes H^0(F,\omega_F^{\otimes k})\right]^G$$

Since non-zero regular 1-form on B are translation-invariant,

$H^0(B,\omega_B^{\otimes k})$ G-invariant.

$$P_k(S)=\dim H^0(F,\omega_F^{\otimes k})^G=\dim H^0(F/G,\mathcal{L}_k)$$

where $\mathcal{L}_k=\omega_{F/G}^{\otimes k}\left(\sum_{P\in F/G}\left[k\left(1-\frac{1}{e_P}\right)\right]P\right)$

Since $F/G\cong\mathbb{P}^1$, $\mathcal{L}_k$ determined by its degree

$$\deg\mathcal{L}_k=-2k+\sum_{P\in F/G}\left[k\left(1-\frac{1}{e_P}\right)\right]$$

Riemann-Hurwitz formula for $F\longrightarrow F/G$

$$\rightsquigarrow 2g(F)-2=-2n+\sum_P n\left(1-\frac{1}{e_P}\right)$$

Note that if $\exists$ r ramification points, then

$$\deg L_k\geq -2k+\sum_P\left(k\left(1-\frac{1}{e_P}\right)-1\right)\overset{\vee}{=}k\frac{2g(F)-2}{n}-r$$

Hence if $g(F) \geq 2$, $P_k(S) = \max\{\deg L_k + 1, 0\} \to +\infty$ as $k \to +\infty$

$\Rightarrow$ (2) holds.

Write the ramification indices in increasing order:

$$e_1 \leq \cdots \leq e_r.$$

R.-H $\rightsquigarrow$ $\sum (1 - \frac{1}{e_i}) \geq 2$.

We must show that $\deg L_k \geq 0$ for some suitable k ($k | 12$)

divide further into following subcases:

(a) $r \geq 4$

since $2(1 - \frac{1}{e_i}) \geq 1$, $\deg L_2 \geq 0$

R.-H. $\rightsquigarrow$ $r \geq 3$.

So <u>WMA</u> $r = 3$

then $\frac{1}{e_1} + \frac{1}{e_2} + \frac{1}{e_3} \leq 1$

(b) <u>$(e_1 \geq 3)$</u>

then $3(1 - \frac{1}{e_i}) \geq 2$ & so $\deg L_3 \geq 0$.

So <u>WMA</u> $e_1 = 2$

then $\frac{1}{e_2} + \frac{1}{e_3} \leq \frac{1}{2}$

(c) $(e_2 \geq 4)$

then $\deg L_4 \geq 0$

(d) $(e_2 = 3)$

then $e_3 \geq 6$ and so $\deg L_6 \geq 0$.

(1) holds in Case I.

Theorem (Enriques)

S surface with $P_4 = P_6 = 0$ (or $P_{12} = 0$)

then S is ruled.

Pf. If $q = 0$, then Castelnuovo's rationality criterion $\Rightarrow$

S rational.

If $q \geq 1$. by above propositions, S ruled.

$\square$

Cor TFAE

(1) S ruled

(2) $\exists$ non-exceptional curve C on S s.t. $K_S C < 0$

(3) for $\forall$ divisor D on S,

$$|D + nK| = \phi \quad \text{for } n \gg 0$$

("adjunction terminates")

(4) $P_n = 0$ for $\forall n$

(5) $P_{12} = 0$.

Pf. "(1) $\Rightarrow$ (2)" By the structure theorem for minimal models of ruled surfaces,

$\exists$ birat'l morph. $f : S \longrightarrow X$

X : geometrically ruled surface or $\mathbb{P}^2$

$\exists$ a fibre F of X or a line L in $\mathbb{P}^2$ over which f is an isom.

then $f^*F . K_S = F . K_X = -2$

or

$$f^*L . K_S = L . K_X = -3$$

"(2) $\Rightarrow$ (3)"

C non-exceptional $\xrightarrow{\text{genus formula}}$ $C^2 \geq 0$

$(D + nK) C < 0$ for all $n \gg 0$

$\Rightarrow$ $|D + nK| = \phi$ for $n \gg 0$

"(3) $\Rightarrow$ (4) $\Rightarrow$ (5)" ✓

"(5) $\Rightarrow$ (1)" by Enriques' theorem.