Lattices and Root Bases

§1 Lattices theory

<u>Def</u> a lattice is a free abelian group M of finite rank $r = \mathrm{rk}\, M$ equipped with a symmetric bilinear form $\varphi : M \times M \longrightarrow \mathbb{Z}$

$$(x, y) \longmapsto x \cdot y$$

$$(x, x) \longmapsto x^2$$

$\varphi \overset{\text{linearity}}{\rightsquigarrow}$ defines a symmetric bilinear form on the $\mathbb{R}$-vector space

$M_{\mathbb{R}} = M \otimes_{\mathbb{Z}} \mathbb{R}$

$\rightsquigarrow$ the corresponding real quadratic form $x \longmapsto x^2$ has signature (t_+, t_-, t_0)

Called the <u>signature</u> of the lattice M.

So we can speak about positive definite, negative definite, semi-definite & indefinite lattices.

A lattice M is called <u>non-degenerate</u> if $t_0 = 0$.

in this case, often write (t_+, t_-) for the signature of M.

- A homomorphism of lattices $f : M \longrightarrow M'$ is a homo. of abelian groups s.t. $f(x) \cdot f(y) = x \cdot y$ for $\forall x, y \in M$.

If f injective, then $f : M \hookrightarrow M'$ called an embedding

If f bijective, then $f : M \longrightarrow M'$ called an isometry.

&

two lattices M and M' are called isomorphic

The set of isometries $\sigma : M \longrightarrow M$ is a group w.r.t. composition

Called the orthogonal group of M

$O(M)$

- A sublattice M' of a lattice M is a subgroup $M' \subset M$ equipped with the induced bilinear form.

a sublattice M' of M is called <u>primitive</u> if the quotient group ~~M/M'~~ M/M' is a free abelian group.

a sublattice $M' \subset M$ is <u>of finite index n</u> if the quotient group M/M' is a finite group of order n.

- An element $m \in M$ is called a <u>primitive element</u> if the sublattice $\mathbb{Z}m \subset M$ is primitive.

- The <u>sum</u> of two sublattices M_1, M_2 of M is the minimal sublattice of M containing M_1 and M_2, denoted by $M_1 + M_2$.

If for $\forall\, m_1 \in M_1,\ m_2 \in M_2$

$$m_1 \cdot m_2 = 0 \in \mathbb{Z}$$

this sum $M_1 + M_2$ is said to be the <u>orthogonal sum</u> of sublattices denoted by $M_1 \oplus M_2$

- The <u>orthogonal sum</u> of two lattices M, M' is the ~~the~~ lattice $M \oplus M'$ equipped with the bilinear form

$$(M \oplus M') \times (M \oplus M') \longrightarrow \mathbb{Z}$$

$$(x, x') \cdot (y, y') := x \cdot y + x' \cdot y'$$

- The <u>orthogonal complement</u> of a sublattice N of a lattice M is the sublattice

$$N^\perp = \{ x \in M \mid x \cdot y = 0 \text{ for } \forall\, y \in N \}$$

- given a lattice M, (M, $\varphi : M \times M \to \mathbb{Z}$ symmetric bilinear)

$M^* := \mathrm{Hom}_{\mathbb{Z}}(M, \mathbb{Z})$ an abelian group

$\rightsquigarrow$ a homomorphism of abelian groups

$$i_M : M \longrightarrow M^*$$
$$x \longmapsto \varphi(x, \cdot)$$

The kernel of i_M is denoted by $\mathrm{Rad}(M)$.

Called the <u>radical</u> of M

Clearly, the lattice M is non-degenerate $\Leftrightarrow \mathrm{Rad}(M) = 0$

The cokernel of i_M is denoted by $\overset{A_M}{\overset{\parallel}{D}}(M) := M^* / i_M(M)$

Called the <u>discriminant group</u> of M

If M nondegenerate, then $\underset{M^*/M}{\underset{\parallel}{D(M)}}$ is a finite abelian group. [$\mathrm{rank}\, M^* = \mathrm{rank}\, M$]

its order called the <u>discriminant of M</u>, denoted by $\mathrm{discr}(M)$

A lattice M is called <u>unimodular</u> if i_M bijective.

If $\underline{e} = (e_1, \cdots, e_r)$ is a $\mathbb{Z}$-basis of M, the matrix

$$G(\underline{e}) = \begin{pmatrix} e_1\cdot e_1 & e_1\cdot e_2 & \cdots & e_1\cdot e_r \\ e_2\cdot e_1 & e_2\cdot e_2 & \cdots & e_2\cdot e_r \\ \vdots & \vdots & \ddots & \vdots \\ e_r\cdot e_1 & e_r\cdot e_2 & \cdots & e_r\cdot e_r \end{pmatrix}_{r\times r}$$

Called the <u>Gram matrix of M</u> w.r.t. $\underline{e}$

If $\underline{h} = (h_1, \cdots, h_r)$ is another $\mathbb{Z}$-basis of M, then $\underline{h} = \underline{e}\,P$ for some invertible matrix $\underset{(p_{ij})}{\overset{}{P}} \in GL(r, \mathbb{Z})$, that is

$$h_i = p_{1i}e_1 + p_{2i}e_2 + \cdots + p_{ri}e_r = (p_{1i} \cdots p_{ri})\begin{pmatrix} e_1 \\ \vdots \\ e_r \end{pmatrix}$$

$$h_j = (e_1 \cdots e_r)\begin{pmatrix} p_{1j} \\ \vdots \\ p_{rj} \end{pmatrix}$$

$$\Rightarrow \quad h_i \cdot h_j = (p_{1i} \cdots p_{ri})\, G(\underline{e}) \begin{pmatrix} p_{1j} \\ \vdots \\ p_{rj} \end{pmatrix}$$

$$G(\underline{h}) = P^T G(\underline{e}) P$$

P invertible in $M_{r\times r}(\mathbb{Z}) \Rightarrow \det(P) = \pm 1$ $\Big\} \Rightarrow \det G(\underline{h}) = \det G(\underline{e})$

i.e. determinant of Gram matrix is independent of the choice of $\mathbb{Z}$-basis of the lattice M.

By Smith Normal Form, if M non-degenerate,

$$\operatorname{discr}(M) = |\det G(\underline{e})| \quad \text{for } \forall\ \mathbb{Z}\text{-basis } \underline{e} \text{ of } M.$$

<u>Rmk</u>: lattice structure on a free abelian group determined by Gram matrix w.r.t. some $\mathbb{Z}$ basis.

- a lattice M is called <u>even</u> if $x^2 \in 2\mathbb{Z}$ for $\forall\, x \in M$

In this case, the map $x \longmapsto x^2$ is an integral quadratic form

that is a free abelian group M together with a map $q: M \to \mathbb{Z}$ such that

- $q(nx) = n^2 q(x)$ for $\forall\, n \in \mathbb{Z}, x \in M$
- the map $$\begin{aligned} M \times M &\longrightarrow \mathbb{Z} \\ (x, y) &\longmapsto q(x+y) - q(x) - q(y) \end{aligned}$$ is a symmetric bilinear form on M.

In this sense

$$\{\text{even lattices}\} \overset{1:1}{\longleftrightarrow} \{\text{integral quadratic forms}\}$$

<u>Some notations</u>

$n \in \mathbb{Z}$ integer, M lattice

$M(n)$: the lattice obtained from the lattice M by multiplying the values of its symmetric bilinear form by n

$\langle n \rangle$: the lattice $\mathbb{Z}x$ of rank 1 with $x^2 = n$

$M_n = \{ x \in M \mid x^2 = n \}$

elements of M_0 called isotropic vectors (迷向向量)

M_n' : subset of primitive vectors from $M_n = \{x \in M \mid x^2 = n\}$

M^n : orthogonal sum of n copies of M

$U_{[n]} = \mathbb{Z} e_1 + \cdots + \mathbb{Z} e_n$

where $e_i \cdot e_j = 1 - \delta_{ij}$

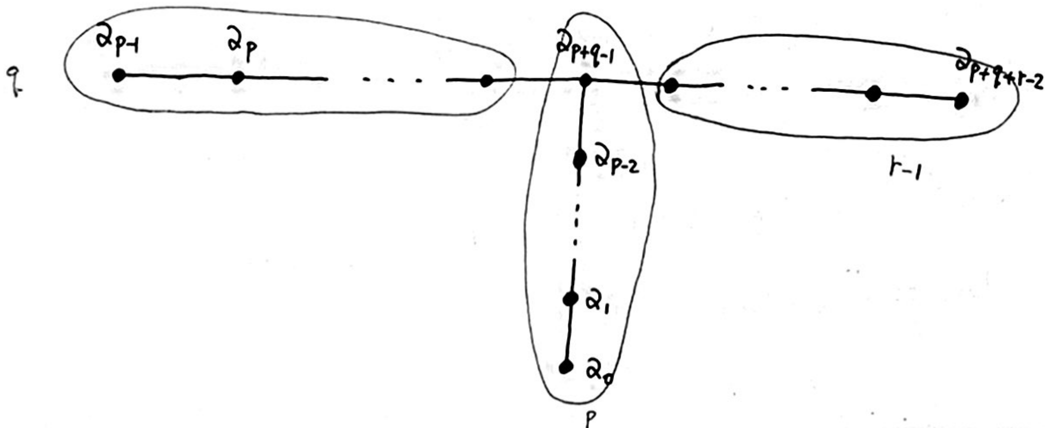
When $n = 2$, this rank 2 indefinite lattice defined by the Gram matrix $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

Called the (standard) hyperbolic plane, denoted by U.

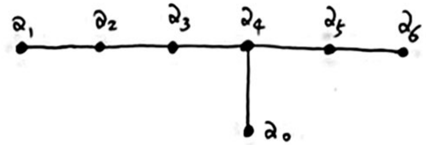
$Q_{p,q,r} = \mathbb{Z} a_0 + \cdots + \mathbb{Z} a_{p+q+r-2}$

where $a_i^2 = -2$, $a_i \cdot a_j = 1$ or 0 depending on whether a_i joined to a_j or not

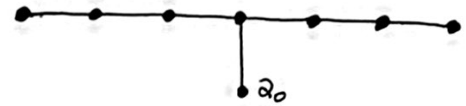
in the following graph $T_{p,q,r}$



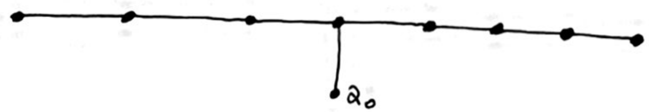
$E_6 = Q_{2,3,3}$



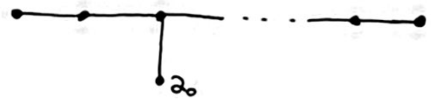
$E_7 = Q_{2,3,4}$



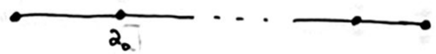
$E_8 = Q_{2,3,5}$



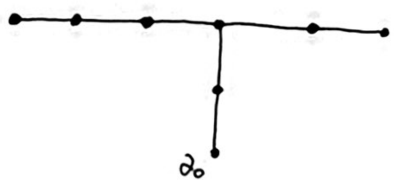
$D_n = Q_{2,2,n-2}$ $(n \geq 4)$



$A_n = Q_{1,1,n}$

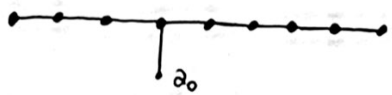


$\tilde{E}_6 = Q_{3,3,3}$



$\tilde{E}_7 = Q_{2,4,4}$

$\tilde{E}_8 = Q_{2,3,6}$



Root bases & Weyl groups

M lattice

Def a non-empty subset B of M_{-2} is called a <u>root basis</u> if

$$\begin{cases} \alpha\cdot\beta \geq 0 \quad \text{for } \forall\ \alpha \neq \beta \in B \\ \alpha \text{ not a linear combination with positive coeff of elements of } B\setminus\{\alpha\} \end{cases}$$

For $\forall\ \alpha \in B$, the map

$$s_\alpha : M \longrightarrow M$$
$$x \longmapsto x + (x\cdot\alpha)\alpha$$

is an isometry of M, called the <u>simple reflection into α</u>.

The subgroup of $O(M)$ generated by all simple reflections s_α is called the <u>Weyl group of B</u>, denoted by $W_B(M)$ (or W)

An element $\alpha' \in M_{-2}$ is called a <u>root</u> w.r.t. B if

$$\alpha' = \theta(\alpha) \quad \text{for some } \theta \in W_B(M)\ \&\ \alpha \in B$$

(i.e. $\alpha' \in W_B$. the orbit of the set B w.r.t. W)

The set of all roots denoted by $R(B)$ ($/R$)

B : ordered finite root basis $\rightsquigarrow$ matrix $(\alpha\cdot\beta)_{\alpha,\beta\in B}$

called the <u>Carton matrix of B</u>

Dynkin diagram of B:

the graph $\Gamma(B)$:
- set of vertices = set B
- two vertices are joined by an edge if $\alpha\cdot\beta \geq 1$
- the edge is labelled by the number $(\alpha\cdot\beta - 1)$ if it is great than 1.

- A root basis B is said to be <u>irreducible</u> if $B \neq B_1 \sqcup B_2$, where B_1, B_2 are root basis s.t. $\alpha\cdot\beta = 0$ for $\forall\ \alpha \in B_1,\ \beta \in B_2$.

B is irreducible $\Longleftrightarrow$ its Dynkin diagram $\Gamma(B)$ is connected.

the basis $B = \{\alpha_0, \ldots, \alpha_{p+q+r-2}\}$ of the lattice $Q_{p,q,r}$ is an irred. root basis

Corresponding graph $T_{p,q,r}$ is the Dynkin diagram of B.

A root basis in $Q_{p,q,r}$ called <u>canonical</u> if its Dynkin diagram is of type $T_{p,q,r}$

$M_{\mathbb{R}} := M \otimes_{\mathbb{Z}} \mathbb{R}$ real inner product vector space

let $C(B) := \{ x \in M_{\mathbb{R}} \mid x \cdot \alpha_i \geq 0 \text{ for } \forall \alpha_i \in B \}$

called the fundamental chamber w.r.t. B &

$$K(B) = \bigcup_{\theta \in W} \theta(C(B))$$

called the Tits cone w.r.t. B.

Def the length function of the Weyl group W_B is the map

$$l = \text{length} : W_B \longrightarrow \mathbb{Z}_{\geq 0}$$

$\theta \longmapsto$ minimal number q s.t. θ can be written as a product of q simple reflections s_α, $\alpha \in B$

Prop For $\forall \alpha \in B$, let $A_\alpha := \{ x \in M_{\mathbb{R}} \mid x \cdot \alpha \geq 0 \}$

let $\theta \in W_B$

then $\theta(C(B)) \subseteq s_\alpha(A_\alpha) \Leftrightarrow \text{length}(s_\alpha \circ \theta) = \text{length}(\theta) - 1$.

Cor For $\forall x \in K(B)$, $\exists\ \theta \in W_B$ s.t. $\theta(x) \in C(B)$ &

$x = \theta(x) + \sum_{\alpha \in B} m_\alpha \alpha$ where $m_\alpha \geq 0$.

Def A root $\alpha \in \underset{\substack{\| \\ R_B}}{R(B)}$ is called positive w.r.t. B if $\alpha \cdot x \geq 0$ for $\forall x \in C(B)$.

$R_B^+ = \{ \text{all positive roots} \}$

Prop $\Rightarrow$ for $\forall$ root $\alpha \in B$, either α or $-\alpha$ is positive.

$$\Rightarrow R_B = R_B^+ \sqcup R_B^-$$

$$R_B^- = \{ -\alpha \mid \alpha \in R_B^+ \}$$

In the case $B = \{ \alpha_1, \cdots, \alpha_r \}$ a basis of $M_{\mathbb{R}}$

$$C(B) = \mathbb{R}_{\geq 0}\, \alpha_1^* + \cdots + \mathbb{R}_{\geq 0}\, \alpha_r^*$$

here $\{ \alpha_1^*, \cdots, \alpha_r^* \}$ dual basis in $(M_{\mathbb{R}})^*$.

In particular,

$$R_B^+ = R_B \cap (\mathbb{Z}_{\geq 0}\, \alpha_1 + \cdots + \mathbb{Z}_{\geq 0}\, \alpha_r)$$

Enriques Surfaces

Def S nonsingular projective surface $/k=\bar{k}$, char $k = p \neq 2$

if $\begin{cases} K_S \underset{num}{\equiv} 0 \text{ but } K_S \underset{lin}{\not\sim} 0 \\ \& \\ b_2(S) = 10 \end{cases}$

then S is called an Enriques surface.

[equivallently,
S minimal, $\kappa(S)=0$ & $b_2(S)=10$]

always work over $k=\bar{k}$, char $k=0$ (by Lefschetz principle, can consider $k=\mathbb{C}$)

Invariants of Enriques surfaces

$\left.\begin{array}{l} K_S \underset{num}{\equiv} 0 \\ \text{but } K_S \underset{lin}{\not\sim} 0 \end{array}\right\} \Rightarrow K_S$ torsion line bundle $\Rightarrow \cancel{h^0(K_S)=}\ h^q(K_S)=0$

i.e. $p_g(S)=0$

$\left.\begin{array}{l} e(S) = 2-2b_1(S)+b_2(S) = 12-4q(S) \\ K_S^2 = 0 \\ 12\chi(\mathcal{O}_S) = K_S^2 + e(S) \end{array}\right\} \Rightarrow \begin{array}{l} q(S)=0 \\ b_1(S)=0 \end{array}$

Summary: $p_g = q = 0$, $\chi(\mathcal{O}_S)=1$

$b_1=0$, $b_2=10$, $e(S)=12$

lemma | S Enriques surface

$\Rightarrow \omega_S^{\otimes 2} \cong \mathcal{O}_S$ (i.e. $2K_S \underset{lin}{\sim} 0$)

Pf Otherwise, if $2K_S \underset{lin}{\not\sim} 0$, $2K_S \underset{num}{\equiv} 0$, $2K_S$ torsion l.b.

then $\left.\begin{array}{l} P_2(S) = h^0(2K_S) = 0 \\ \text{also } q(S) = 0 \end{array}\right\} \xRightarrow{\text{Castelnuovo's rationality}} S$ rational

$\lightning$

$\square$

Cor | S Enriques Surface

$\mathrm{Pic}^\tau(S) \cong \mathbb{Z}/2$ (generated by K_S)

pf let $L \in \mathrm{Pic}^\tau(S)$ (i.e. a torsion line bundle on S)

then R.-R. $\rightsquigarrow$

$h^0(L) + h^2(L) \geq 1$

$\Rightarrow L \cong \mathcal{O}_S$ or $L \cong \underset{\substack{\uparrow \\ \text{torsion } 2}}{K_S}$

$\square$

lemma | S Enriques surface

$\Rightarrow \rho(S) = b_2(S) = 10$ & $H^1(S,\mathbb{Z}) = 0$

pf Consider the exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_S \xrightarrow{\exp} \mathcal{O}_S^\times \to 0$$

taking cohomology

$$0 \to H^1(S,\mathbb{Z}) \to \underset{\underset{0}{\parallel}}{H^1(\mathcal{O}_S)} \to \operatorname{Pic} S \xrightarrow{c_1} H^2(S,\mathbb{Z}) \to \underset{\underset{0}{\parallel}}{H^2(\mathcal{O}_S)}$$

$\Rightarrow H^1(S,\mathbb{Z}) = 0$ & $\operatorname{Pic} S \cong H^2(S,\mathbb{Z})$

Cor | S Enriques surfaces

$\Rightarrow \underset{\underset{\operatorname{Pic} S}{\parallel}}{H^2(S,\mathbb{Z})} \cong \mathbb{Z}^{10} \oplus \mathbb{Z}/2$

K3 - cover

S : Enriques surface $\rightsquigarrow \omega_S \cong \mathcal{O}_S(K_S)$ torsion-2 line bundle

fix an isom. $\omega_S^{\otimes 2} \cong \mathcal{O}_S \overset{\text{define}}{\rightsquigarrow}$ an $\mathcal{O}_S$-algebra $\mathcal{O}_S \oplus \omega_S$

$\rightsquigarrow$ finite flat double cover

$$\pi : X := \operatorname{Spec}_{\mathcal{O}_S}(\mathcal{O}_S \oplus \omega_S) \longrightarrow S$$

which is étale (double cover)

$$\left.\begin{aligned} &\Rightarrow \omega_X \cong \pi^*(\omega_S \otimes \omega_S^\vee) = \pi^*\mathcal{O}_S \cong \mathcal{O}_X \\ &\chi(\mathcal{O}_X) = 2\chi(\mathcal{O}_S) = 2 \end{aligned}\right\} \Rightarrow \begin{aligned} &K_X \overset{\text{lin}}{\sim} 0 \\ &p_g(X) = 1 \\ &q(X) = 0 \end{aligned}$$

i.e. X K3 surface

Conversely, if X is a K3 surface & ι a fixed-point free involution

then $S := X/\iota$ is an Enriques surface.

Indeed, $\pi : X \longrightarrow \underset{\underset{\text{smooth}}{\uparrow}}{S := X/\iota}$ is an étale double cover

$$\left.\begin{aligned} &\pi^*\omega_S \cong \omega_X \cong \mathcal{O}_X \\ &\cancel{\omega_S^{\otimes 2} \cong \pi_*(\pi^*\omega_S) = \omega_S \otimes \pi_*\mathcal{O}_X} \\ &\pi_*\mathcal{O}_X \cong \pi_*(\pi^*\omega_S) \cong \omega_S \otimes \pi_*\mathcal{O}_X \end{aligned}\right\} \Rightarrow \omega_S^{\otimes 2} \cong \mathcal{O}_S$$

$$\left.\begin{aligned} &\overset{\overset{24}{\parallel}}{e(X)} = 2e(S) \Rightarrow e(S) = 12 \\ &q(X) = 0 \rightarrow q(S) = 0 \\ &\chi(\mathcal{O}_X) = 2\chi(\mathcal{O}_S) \Rightarrow \chi(\mathcal{O}_S) = 1 \end{aligned}\right\} \Rightarrow p_g(S) =$$

$\Downarrow$

$\omega_S \not\cong \mathcal{O}_S$

S Enriques surface

linear systems on Enriques surfaces

<u>Theorem</u> | D an ample divisor on an Enriques surface S

then $\exists$ irreducible curve $C \in |D|$.

<u>pf.</u> $D^2 > 0 \Rightarrow \dim|D| > 0$

$|D| = |M| + Z$, where M : moving part of $|D|$

Z : fixed part

<u>Case 1</u> $M^2 = 0$

then $|M|$ is composed of a genus 1 pencil & $DM \geq 2$

$\Rightarrow D^2 = DM + DZ > 2$ ↯

<u>Case 2</u> $M^2 > 0$ & $|M|$ not composed with a pencil.

If $Z \neq 0$,

$$\dim|M| = \frac{1}{2}M^2$$

$$\|$$

$$\dim|M+Z| \geq \frac{1}{2}(M+Z)^2 = \frac{1}{2}M^2 + \frac{1}{2}(2MZ + Z^2)$$

$\Rightarrow 2MZ + Z^2 \leq 0$

$\Rightarrow DZ = MZ + Z^2 = (2MZ + Z^2) - MZ \leq 0$ ↯.

<u>THEOREM</u> | S Enriques surface

$C \subset S$ irreducible curve with $C^2 \geq 0$, $\dim|C| > 0$

then general member of $|C|$ is smooth.