Algebraic K3 surfaces

- k arbitrary field

a Variety $/k$ is a separated, geometrically integral scheme of finite type $/k$

Def an (algebraic) K3 surface over k is a complete non-singular algebraic surface S s.t.

$$\omega_S \cong \mathcal{O}_S \ \& \ H^1(S, \mathcal{O}_S) = 0$$

here a variety X/k is complete if $X \longrightarrow \mathrm{Spec}\, k$ is proper

nonsingular if cotangent sheaf $\Omega^1_{X/k}$ is locally free of rank $\dim X$

Rmk By definition, for a K3 surface S, Ω^1_S is locally free of rank 2.

The natural alternating pairing

$$\Omega^1_S \times \Omega^1_S \longrightarrow \Omega^2_S \cong \omega_S \cong \mathcal{O}_S$$

$\rightsquigarrow$ a non-canonical isom. $\mathcal{H}om(\Omega^1_S, \mathcal{O}_S) \cong \Omega^1_S$

i.e. $\underset{\text{tangent sheaf}}{\mathcal{T}_S} \cong \Omega^1_S$

Fact any smooth complete surface $/_{k=\bar{k}}$ is projective.

[Zariski-Goodman theorem : X complete nonsingular surface & $U \subseteq X$ a nonempty affine open subset

$\Rightarrow$ $X \setminus U$ is a connected alg subset of pure codim 1 in X, supporting on an ample effective divisor D on X, that is $\mathrm{Supp}\, D = X - U$

In particular, X is projective.]

Rmk All algebraic K3 surfaces are projective.

Complex K3 Surfaces

Def | A complex K3 surface is a compact connected complex manifold X of dim 2 s.t. $\Omega_X^2 \cong \mathcal{O}_X$ & $H^1(X, \mathcal{O}_X) = 0$.

Prop | If X is an algebraic K3 surface over $k = \mathbb{C}$ then the associated complex space X^{an} is a complex K3 surface.

This follows from Serre's GAGA principle

X scheme of finite type$/\mathbb{C}$ $\rightsquigarrow$ a complex space X^{an}

$X^{an} \longrightarrow X$ a morph. of ringed spaces & underlying set of points is just the set of all closed points of X

$\mathcal{F}$ coherent sheaf on X $\rightsquigarrow$ $\mathcal{F}^{an}$ coherent sheaf on X^{an}

s.t.
- $\mathcal{O}_X^{an} \simeq \mathcal{O}_{X^{an}}$
- $\Omega_{X/\mathbb{C}}^{an} \simeq \Omega_{X^{an}}$
- $H^i(X, \mathcal{F}) \cong H^i(X^{an}, \mathcal{F}^{an})$ for all $i \geq 0$

In this sense

$$\begin{tikzcd}
\{\text{algebraic K3 surfaces}/\mathbb{C}\} \arrow[r, hook] \arrow[rd, mapsto] & \{\text{Complex K3 surfaces}\} \\
 & \{\text{Complex projective K3s}\} \arrow[u, phantom, "X\cup" description]
\end{tikzcd}$$

Fact All complex K3 surfaces are kähler [Siu'1983 Invent. Math.]

Fact | $\exists$ non-projective complex K3 surfaces. For example

$$\begin{tikzcd}
X \arrow[r, "\text{minimal}", "\text{resolution}"'] & A/\iota \\
 & A = \mathbb{C}^2/\Gamma \arrow[u]
\end{tikzcd}$$

$A = \mathbb{C}^2/\Gamma$ | complex torus, non-projective

ι : involution of A

For connected compact complex manifold X of dim n

$H^{p,q}(X) \cong H^q(X, \Omega_X^p)$ Dolbeault's isom.

If X compact kähler mfd. then we have Hodge deco

$$H^k(X, \mathbb{C}) \cong \bigoplus_{p+q=k} H^{p,q}(X) \quad \& \quad h^{p,q} = h^{q,p}$$

- For all complex k3 surface S, from

$$\begin{tikzcd}[column sep=small] H^1(S,\mathbb{C}) \arrow[r, phantom, "\cong"] & H^1(S,\mathcal{O}_S) \arrow[r, phantom, "\oplus"] \arrow[d, phantom, "\wr\wr"] & H^0(S,\Omega^1_S) \arrow[d, phantom, "\wr\wr"] \\ & H^{0,1} & H^{1,0} \end{tikzcd}$$

$$\Rightarrow H^0(S,\Omega^1_S)=0 \quad \& \quad H^1(S,\mathbb{C})=0$$

$$b_1(S)=0$$

$\uparrow$ no non-trivial global vector fields.

- From the exponential sequence

$$0 \to \mathbb{Z} \to \mathcal{O}_S \to \mathcal{O}_S^* \to 0$$

taking cohomology.

$$\begin{tikzcd} & & & \operatorname{Pic} S \arrow[d, phantom, "\wr\wr"] & \\ 0 \arrow[r] & H^1(S,\mathbb{Z}) \arrow[r] & H^1(S,\mathcal{O}_S) \arrow[r] & H^1(S,\mathcal{O}_S^*) \arrow[r] & H^2(S,\mathbb{Z}) \arrow[d] \\ & 0 & H^3(S,\mathbb{Z}) \arrow[l] & H^2(S,\mathcal{O}_S^*) \arrow[l] & H^2(S,\mathcal{O}_S) \arrow[l] \end{tikzcd}$$

S complex k3 $\Rightarrow H^1(S,\mathcal{O}_S)=0 \Rightarrow \boxed{H^1(S,\mathbb{Z})=0}$

$\Downarrow$

$H^0(S,\mathbb{Z}) \simeq H^4(S,\mathbb{Z}) \cong \mathbb{Z}$

$\Downarrow$ Poincaré duality

$H^3(S,\mathbb{Z})=0$ (up to torsion)

- Serre duality $H^2(S,\mathcal{O}_S) \cong H^0(S,\omega_S)$

$$\begin{tikzcd} H^2(S,\mathcal{O}_S) \arrow[r, phantom, "\cong"] & H^0(S,\omega_S) \arrow[d, phantom, "\wr\wr"] \\ \boxed{p_g(S)=1} & H^0(S,\mathcal{O}_S)\cong\mathbb{C} \end{tikzcd}$$

- Fact from topology

$$(\text{torsion of } H^i(X,\mathbb{Z})) \overset{\text{identified}}{\sim} (\text{torsion of } H^{\dim_{\mathbb{R}} X - i+1}(X,\mathbb{Z}))$$

- $\operatorname{Pic}(S)$ has no torsion line bundles for k3 S

$\Rightarrow H^2(S,\mathbb{Z})$ no torsion $\Rightarrow H^3(S,\mathbb{Z})$ no torsion & hence $H^3(S,\mathbb{Z})=0$

By Noether formula

$$\left.\begin{array}{l} \left(\begin{array}{l} \chi(\mathcal{O}_S)=1-q(S)+p_g(S)=2 \\ K_S^2=0 \end{array}\right) \\ 12\chi(\mathcal{O}_S)=K_S^2+e(S) \end{array}\right\} \Rightarrow e(S)=24$$

$$e(S)=2-2b_1+b_2 \Rightarrow b_2=22 \Rightarrow h^{1,1}=20$$

✡ $H^2(S,\mathbb{Z})$ free abelian group of rank 22

Hodge diamond for K3

$$\begin{array}{ccccccccccc}
 & & h^{0,0} & & & b_0 & & & 1 & & \\
 & h^{1.0} & & h^{0,1} & & b_1 & & 0 & & 0 & \\
h^{2.0} & & h^{1,1} & & h^{0.2} & b_2 & 1 & & 20 & & 1 \\
 & h^{2,1} & & h^{1.2} & & b_3 & & 0 & & 0 & \\
 & & h^{2,2} & & & b_4 & & & 1 & &
\end{array}$$

K3 : named in honor of Kummer, Kähler, Kodaira.

In the following, we will only consider algebraic K3 surface $/\mathbb{C}$, so all are projective.

Smooth curves on K3 surfaces

Lemma

S : K3 surface

$C \subset S$: smooth irreducible curve of genus g

$L := \mathcal{O}_S(C)$

then $L^2 = 2g-2$ & $h^0(L) = g+1$

proof. By adjunction formula / genus formula,

$$\underset{\underset{2g-2}{\shortparallel}}{2p_a(C)-2} = C^2 + K_S C = L^2$$

By Riemann-Roch,

$$\underset{\underset{h^0(L)-h^1(L)}{\shortparallel}}{h^0(L)-h^1(L)+h^0(L^{-1})} = \chi(L) = \underset{\underset{\underset{\underset{g+1}{\shortparallel}}{2+\frac{1}{2}L^2}}{\shortparallel}}{\chi(\mathcal{O}_S)+\frac{1}{2}(L^2-LK_S)}$$

$$\Rightarrow h^0(L) \geq g+1$$

Adjunction formula $\omega_C \cong (K_S+C)|_C \cong L|_C$

$$0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0$$

$$0 \to \mathcal{O}_S \to \underset{\underset{L}{\shortparallel}}{\mathcal{O}_S(C)} \to L|_C \to 0$$

$$\rightsquigarrow 0 \to H^0(\mathcal{O}_S) \to H^0(S,L) \to \underset{\underset{H^0(C,\omega_C)}{\wr\wr}}{H^0(C,L|_C)} \to \overset{\overset{0}{\shortparallel}}{H^1(\mathcal{O}_S)}$$

$$\Rightarrow \underset{\underset{g}{\shortparallel}}{h^0(C,\omega_C)} \geq h^0(L)-1$$

$$\Rightarrow \begin{cases} h^0(L) = g+1 \text{ \& } H^0(S,L) \twoheadrightarrow H^0(C,L|_C) \\ h^1(L) = 0 \end{cases} \square$$

Rmk. The same line proof $\rightsquigarrow H^0(S,L^{\otimes m}) \twoheadrightarrow H^0(C,L^{\otimes m}|_C)$ for all $m>0$

Indeed,

$$h^0(S,L^{\otimes m}) \geq \chi(L^{\otimes m}) = 2+\frac{1}{2}(mL)^2 = 2+\frac{m^2}{2}L^2$$

$$0 \to L^{\otimes(m-1)} \to L^{\otimes m} \to L^{\otimes m}|_C \to 0 \quad \text{exact}$$

$$\rightsquigarrow 0 \to H^0(L^{\otimes(m-1)}) \to H^0(L^{\otimes m}) \to H^0(C,L^{\otimes m}|_C)$$

$$\Rightarrow h^0(L^m) \leq h^0(C,L^m|_C)+h^0(L^{m-1}) \leq \underset{\underset{\frac{m^2}{2}L^2+2}{\shortparallel}}{m\deg L|_C+1-g(C)+\frac{(m-1)^2}{2}L}$$

$$\Rightarrow h^0(L^m) = 2+\frac{m^2}{2}L^2 \text{ \& } \underset{\underset{\text{Surjectivity}}{\Downarrow}}{h^0(L^m) = h^0(L^{m-1})+h^0(L^m|_C)} \quad \square$$

Line bundles on K3 surfaces.

S : algebraic K3 surface $/\mathbb{C}$
(called K3 surface for short)

L : line bundle on S

Lemma

① If $L^2 \geqslant -2$, then $H^0(S, L) \neq 0$ or $H^0(L^{-1}) \neq 0$

② If $L^2 \geqslant 0$, then either $L \cong \mathcal{O}_S$

or $h^0(L) \geqslant 2$

or $h^0(L^{-1}) \geqslant 2$.

③ If $h^0(L) = 1$ & $\overset{\mathrm{div}(t)}{\overset{\shortparallel}{D}} \geqslant 0$ for $0 \neq t \in H^0(L)$, then for $\forall$ curve $C \subset D$, $C^2 \leqslant -2$

if C integral, then C (-2)-curve.

Pf By Riemann-Roch

$$\begin{array}{ccc} \chi(L) & = & \chi(\mathcal{O}_S) + \frac{1}{2}(L^2 - L.k_S) \\ \shortparallel & & \shortparallel \\ h^0(L) + h^0(L^{-1}) - h^1(L) & & 2 + \frac{1}{2}L^2 \end{array}$$

□

Prop L line bundle on a K3 surface S

then L ample $\Longleftrightarrow$ L is in the positive cone $\mathcal{C} \subset NS(S)_{\mathbb{R}}$

&

$L.C > 0$ for $\forall$ smooth rat'l curve $C \subset S$

Cor If L line bundle on K3 S satisfying $\begin{cases} L^2 \geqslant 0 \\ LC \geqslant 0 \end{cases}$ for $\forall$ smooth rat'l curve $C \subset S$

then L nef unless there is no such curves C, in which case L or L^{-1} nef

key point : L nef $\Longleftrightarrow$ for a fixed ample line bundle H

$\varepsilon L + H$ ample for $\forall \varepsilon > 0$.

Lemma Big and nef curves are 1-connected.

$\uparrow$

$(L^2 > 0)$

- Hodge index theorem :

the signature of the intersection form on $\mathrm{Num}(S) = \mathrm{Pic}\,S \big/ \mathrm{Pic}^{\tau} S$ is $(1, \rho(S)-1)$.

num. trivial l.b.

The cone of all classes $L \in NS(S)_{\mathbb{R}}$ with $L^2 > 0$ has two connected components, the positive cone $\mathcal{C}_S \subset NS(S)_{\mathbb{R}}$ defined as the connected component containing an ample line bundle.

(closed under positive scaler multiplication)

Prop L line bundle on K3 S

$\Rightarrow$ L ample $\Leftrightarrow$ $\begin{cases} L \in \mathcal{C}_S \subset NS(S)_{\mathbb{R}} \\ LC > 0 \text{ for } \forall \text{ smooth rat'l curve } \underset{\cong}{\overset{\mathbb{P}^1}{C}} \subset S \end{cases}$

positive cone

Pf "$\Rightarrow$" ✓

"$\Leftarrow$" For $\forall$ curve $C \subset S$ with $C^2 \geq 0$

$C \in \overline{\mathcal{C}_S}$ (closure of $\mathcal{C}_S$ in $NS(S)_{\mathbb{R}}$)

For ample line bundle H, $HC > 0$

$L \in \mathcal{C}_S$ ~~$\Rightarrow$~~

$\mathcal{C}_S$ connected

$\Rightarrow LM > 0$ for $\forall_{\neq 0} M \in \overline{\mathcal{C}_s}$

$\Downarrow$

$LC > 0$ for $\forall$ curve $C \subset S$ with $C^2 \geq 0$

$2p_a(C) - 2 = K_S C + C^2 = C^2$

If $C^2 < 0$ $\Rightarrow$ C smooth rational curve

$\Downarrow$

$LC > 0$ by assumption

$\Rightarrow$ $\begin{cases} L^2 > 0 \\ LC > 0 \text{ for } \forall \text{ curve } C \subset S \end{cases}$ $\Rightarrow$ L ample.

lemma For smooth irreducible curve $C \subset S$ of genus $g \geq 2$ the line bundle $L = \mathcal{O}_S(C)$ is base-point-free & the induced morphism $\varphi_L : S \longrightarrow \mathbb{P}^{g}$ restricts to the canonical map $C \longrightarrow \mathbb{P}^{g-1}$

Pf. $H^0(S, L) \twoheadrightarrow H^0(C, L|_C) \cong H^0(C, \omega_C)$

$$\omega_C \cong (K_S + C)|_C = L|_C$$

$$\rightsquigarrow \quad H^0(C, \omega_C)^\vee \hookrightarrow H^0(S, L)^\vee$$

$$\mathbb{P}^{g-1} \cong \mathbb{P}(H^0(C, \omega_C))^\vee \hookrightarrow \mathbb{P}(H^0(S, L)^\vee) \cong \mathbb{P}^g$$

$$\left.\begin{array}{l} L \text{ base-point-free outside } C \\ \deg \omega_C = 2g-2 \\ g \geq 2, \ |\overset{L|_C}{\overset{\shortparallel}{\omega_C}}| \text{ no base points} \end{array}\right\} \Rightarrow L \text{ bpf } \& \ \varphi_L|_C = \varphi_{K_C}.$$

Fact $g \geq 2$, $|K_C|$ bpf.

For $\forall\, p \in C$, suffices to show $\dim |K_C - p| = \underbrace{\dim |K_C| - 1}_{\overset{\shortparallel}{g-2}}$

C not rational, $h^0(p) - h^1(p) = 1 + 1 - g$

$\Downarrow$

$h^0(p) = 1$ $\quad \Rightarrow \quad h^0(K_C - p) = g-2$

$\square$

Rmks

(1) If $g = 2$, then the curve C is hyperelliptic & hence $\varphi_L : S \longrightarrow \mathbb{P}^{g=2}$ restricts to a morphism $\varphi_C : C \longrightarrow \mathbb{P}^1$ of degree 2.

$$L = \varphi_L^* \mathcal{O}_{\mathbb{P}^2}(1) \Rightarrow \underset{\underset{\underset{2}{\shortparallel}}{\overset{\shortparallel}{2g-2}}}{L^2} = \deg \varphi_L \Rightarrow \deg \varphi_L = 2$$

$\varphi_L : S \longrightarrow \mathbb{P}^2$ generically double cover of $\mathbb{P}^2$.

(2) For $g \geq 2$, φ_L can be of degree 1 or 2.

$\Rightarrow$ φ_L is birational depending on whether the generic curve C in $|L|$ is hyperelliptic or not.

Accordingly, Calls L hyperelliptic or non-hyperelliptic.

Prop (Projective normality)

S : K3 surface

$C \subset S$: Smooth, irreducible non-hyperelliptic genus g with $g > 2$

$L = \mathcal{O}_S(C)$

$\Rightarrow$ $|L|$ is projectively normal, that is, for $\forall m \geq 0$ the pull-back under φ_L defines a surjective map

$$H^0(\mathbb{P}^g, \mathcal{O}_{\mathbb{P}^g}(m)) \twoheadrightarrow H^0(S, L^m)$$

pf. Consider the s.e.s.

$$0 \to \mathcal{O}_S \to L \to L|_C \to 0$$

$$\begin{tikzcd} 0 \arrow[r] & L^m \arrow[r] & L^{m+1} \arrow[r] & L^{m+1}|_C \arrow[r] \arrow[d, equal] & 0 \\ & & & \omega_C^{m+1} & \end{tikzcd}$$

taking cohomology.

$$H^0(S, L^{m+1}) \to H^0(C, \omega_C^{m+1}) \to H^1(L^m)$$

We have seen that $H^0(L^{m+1}) \twoheadrightarrow H^0(C, \omega_C^{m+1})$

$$\Rightarrow \begin{tikzcd} H^0(\mathbb{P}^g, \mathcal{O}(m+1)) \simeq S^{m+1} H^0(S, L) \arrow[r, two heads] \arrow[dr, "\text{Surjective}"'] & H^0(S, L^{m+1}) \arrow[d] \\ & H^0(C, \omega_C^{m+1}) \end{tikzcd}$$

The kernel of $H^0(S, L^{m+1}) \to H^0(C, \omega_C^{m+1})$ spanned by $t \cdot H^0(S, L^{m+1})$ where t is the section defining C.

By induction hypothesis, $S^m H^0(S, L) \twoheadrightarrow H^0(S, L^m)$

□

Lemma $C \subset S$ Smooth irreducible curve on K3 S

$L = \mathcal{O}_S(C)$

$\Rightarrow$ For $(m \geq 2, g > 2)$ or $(m \geq 3, g = 2)$

the morphism $\varphi_{L^m} : S \longrightarrow \mathbb{P}^{m^2(g-1)+1}$

is birational onto its image.

pf. $\deg \omega_C^m = m(2g-2) \overset{\substack{m\geq 3 \\ g=2}}{=} 2m \geq 2\overset{5}{\overset{\shortparallel}{g+1}} \Rightarrow \varphi_{\omega_C^m}$ embedding.

□

Theorem | L ample l.b. on K3 surface S

$\Rightarrow L^m$ globally generated for $m \geq 2$

very ample for $m \geq 3$

"Pf" Assume $\exists$ smooth curve $C \in |L|$.

then C is irreducible.

L ample $\Rightarrow L^2 > 0 \Rightarrow g(C) \geq 2$

$$\Downarrow$$

$|L|$ bpf i.e. gg

$$\Downarrow$$

L^m gg

Suff. to show L^m very ample for $m=3$

by previous lemma, L^m defines birat'l morph. for $m \geq 3$

$\Rightarrow \varphi_{L^3} : S \longrightarrow \bar{S} \subset \mathbb{P}^{9(g-1)+1}$ birat'l

$\bar{S} = \varphi(S)$

with $L^3 = \varphi^* \mathcal{O}(1)$ ample

If $\bar{S}$ normal, then $\varphi_* \mathcal{O}_S = \mathcal{O}_{\bar{S}}$ by Zariski Main Theorem

$$\Downarrow$$

$H^0(\bar{S}, \mathcal{O}(m)) = H^0(S, L^{3m})$ for $\forall m \geq 1$

L^{3m} very ample for $m \gg 0 \Rightarrow \varphi$ isom.

Claim: $\bar{S}$ normal.

need to find sm curve $D \in |L^3|$ close to $3C$.

φ isom. along D. $\rightsquigarrow$ non-hyperelliptic

by lem., $H^0(\bar{S}, \mathcal{O}(m)) \twoheadrightarrow H^0(S, L^{3m})$

Existence of K3s

Def (polarized K3)

> (S, L)
>
> a polarized K3 surface of degree 2d is a projective K3 surface S together with an ample line bundle L such that L is <u>primitive</u> & $L^2 = 2d$.

(i.e. indivisible in Pic(S)
$L \not\cong M^{\otimes k}$ for some integer $k > 1$ & $M \in \mathrm{Pic}\,S$
cannot be written as a nontrivial tensor power of another l.b.)

Called quasi-polarized K3 (S, L) if L | big & nef
primitive & $L^2 = 2d$

For ∀ smooth curve $C \in |L|$ of genus g

$$g(C) = 1 + \frac{1}{2}(C^2 + K_S C) = 1 + \frac{1}{2}L^2$$

$$\Rightarrow \underset{\substack{\parallel \\ 2d}}{L^2} = 2g - 2$$

often say (S, L) a polarized K3 surface of genus $\overset{\substack{L^2 = 2g-2 \\ \downarrow}}{g}$

THEOREM

> For any integer $g \geq 3$, ∃ K3 surfaces $S = S_{2g-2} \hookrightarrow \mathbb{P}^g$ embedded in $\mathbb{P}^g$.

Pf. Suffices to construct K3 surfaces S containing a very ample divisor D with $D^2 = 2g-2$.

Observation : | H a ~~hyperelliptic~~ hyperplane section of S
| $|M|$ linear system without base points
| $\Rightarrow$ H + M very ample divisor

Indeed, $|H+M|$ separates points : for any two distinct points $x, y \in S$

- ∃ divisor $M' \in |M|$ not containg either x or y
  (such a M' exist sine |M| bpf
- ∃ divisor $H' \in |H|$ containing x but not y
  (since |H| bpf)

$\Rightarrow$ the divisor $H' + M' \in |H+M|$ with

$$x \in H' + M' \text{ but } y \notin H' + M'$$

$|H+M|$ separates tangent vectors : given a point $x \in S$ & tangent vecto
$v \in T_xS = (\mathfrak{m}_x/\mathfrak{m}_x^2)^\vee \Rightarrow \left.\begin{array}{l} \exists\ M' \in |M| \text{ s.t. } x \notin M' \\ \exists\ H' \in |H| \text{ s.t. } x \in H \ \&\ x \notin \cancel{S}\, T_x H \end{array}\right\} \Rightarrow H' + M'$

We now distinguish 3 cases

Case 1 : $g = 3m$ (with $m \geqslant 1$)

let $S \subset \mathbb{P}^3$ be a quartic containing a line ℓ.

let $|E|$ be the pencil of elliptic curves $|H - \ell|$

H a hyperplanes section of $S \subset \mathbb{P}^3$

then

$g = \frac{(d-1)(d-2)}{2}$

$D_m := H + (m-1)E$ very ample

$$D_m^2 = H^2 + 2(m-1)HE + (m-1)^2E^2$$

$$\left.\begin{array}{l} g(E)=1 \Rightarrow E^2 = 0 \\ E \text{ cubic curve} \Rightarrow HE = 3 \\ H^2 = \deg S = 4 \end{array}\right\} \Rightarrow \begin{array}{l} D_m^2 = 4 + 6(m-1) \\ \quad = 2g-2. \end{array}$$

Case 2 : $g = 3m+1$ (with $m \geqslant 1$).

let $Q \subset \mathbb{P}^4$ be a quadric with an ordinary double point.

(i.e. Q is the cone over a smooth quadric in $\mathbb{P}^3$).

let $\widetilde{H} \in |\mathcal{O}_{\mathbb{P}^4}(3)|$ be a cubic s.t. $S = \widehat{H} \cap Q$ smooth

Consider a pencil of planes on Q : it cuts out on $\widetilde{H}$ a pencil of elliptic curves $\underset{\text{Cubics}}{|E|}$ on $Q \cap \widetilde{H} = S$

$\Rightarrow$ $D_m := H + (m-1)E$ very ample on S

$$D_m^2 = H^2 + 2(m-1)HE = \underset{\underset{2g-2}{\shortparallel}}{6 + 6(m-1)}$$

Case 3 : $g = 3m+2$ (with $m \geqslant 1$)

- $S \subset \mathbb{P}^3$ a smooth quartic containing a line ℓ & a twisted cubic t (disjoint from ℓ)
- $E = H - \ell$, $H' = 2H - t$

$(H')^2 = 2 \Rightarrow |H'|$ defines a double cover $\pi : S \longrightarrow \mathbb{P}^2$ branched along a sextic of $\mathbb{P}^2$.

Can prove that $D_m = H' + mE$ very ample for $\forall\, m \geqslant 1$

then $D_m^2 = 2 + 6m = 2g-2$.

$S_{d_1,d_2,\cdots,d_r}$

More examples on k3s

lemma $\overset{\shortparallel}{S} \subset \mathbb{P}^{r+2}$ a surface that is the complete intersection of hypersurfaces $H_1,\cdots,H_r$ of deg $d_1,\cdots,d_r$. resp.
then $K_S \cong \mathcal{O}_S(\sum_{i=1}^{r} d_i - r - 3)$

Pf. By adjunction formula

$$\omega_S \cong \omega_{\mathbb{P}^{r+2}} \otimes \mathcal{O}_{\mathbb{P}^{r+2}}(\Sigma d_i)\big|_S \cong \mathcal{O}_S(\Sigma d_i - r - 3)$$

lemma $X \subset \mathbb{P}^n$ d-dim'l complete intersection, then $H^i(X, \mathcal{O}_X) = 0$ for $0<i<d$.

(①) Complete intersection k3s

$$S_4 \subset \mathbb{P}^3, \quad S_{2,3} \subset \mathbb{P}^4, \quad S_{2,2,2} \subset \mathbb{P}^5$$

(②) double planes (i.e. birat'l isomaphic to an affine surface in $\mathbb{A}^2_{\mathbb{C}}$ $\operatorname{Spec}\frac{k[x,y,z]}{(z^2-f(x,y))}$ with equation $z^2 = f(x,y)$)

Enriques-Campedelli theorem

a double plane $\overset{\text{bir.}}{\simeq}$ k3 is exactly one of the following:

① $z^2 = f_6(x,y)$, where $f_6(x,y)=0$ is a curve of deg 6

② $z^2 = f_8(x,y)$, where $f_8(x,y)=0$ curve of deg 8 having two ordinary quadruple points.

③ $z^2 = f_{10}(x,y)$, where $f_{10}(x,y)=0$ curve of deg 10 having a singular point of multiplicity 7
& two ordinary triple points (infinitely near to 1st order)

④ $z^2 = f_{12}(x,y)$, where $f_{12}(x,y)=0$ a curve of deg 12 having a singular point of multiplicity 9
& 3 ordinary points of multiplicity 3 (infinitely near to 1st order)

(3) triple covers

$$\begin{tikzcd} X \arrow[r, "3:1"] & \mathbb{P}^1\times\mathbb{P}^1 \\ & C \arrow[u, phantom, "\cup"] \end{tikzcd}$$

non-hyperelliptic of genus 4

Such a cover locally defined by the equation

$$z^3 = f(x_1, y_1, x_2, y_2)$$

$f(x_1,y_1,x_2,y_2) \in H^0(\mathbb{P}^1\times\mathbb{P}^1, \mathcal{O}(3,3))$ bihomogeneous

poly of bidegree (3,3) defining C

$\pi: X \to Y$ finite surjective of sm surf ramification index r, branch locus $B \subset Y$

$$\Rightarrow \begin{cases} e(X) = r\,e(Y) - (r-1)\,e(B) \\ K_X = \pi^*(K_Y + \frac{r-1}{r}B) \\ q(Y)=0 \Rightarrow q(X)=0 \quad \text{by Leray Spectral seq.} \end{cases}$$

(4) Kummer surfaces

Smooth minimal models of quotient of an abelian surface A by an involution $x \overset{\tau}{\longmapsto} -x$

A : abelian surface

$\tau : A \longrightarrow A$ involution with 16 points of order 2 as τ fixed points $p_1, \cdots, p_{16}$
$\quad x \longmapsto -x$

$\varepsilon : \hat{A} \longrightarrow A$ blow-up of these 16 points.

$E_i = \varepsilon^{-1}(p_i)$

$\tau \curvearrowright A \rightsquigarrow$ involution $\sigma \curvearrowright \hat{A}$

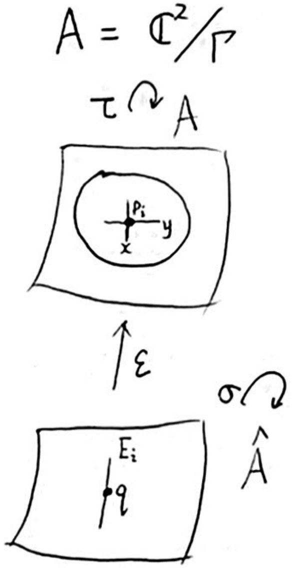
$$X := \hat{A}/\langle\sigma\rangle$$

$$\begin{tikzcd} \hat{A} \arrow[r, "\varepsilon"] \arrow[d, "\pi"'] \arrow[dr, phantom, "\circlearrowleft"] & A \arrow[d] \\ X := \hat{A}/\langle\sigma\rangle \arrow[r] & A/\langle\tau\rangle \end{tikzcd}$$

Prop | $X = \hat{A}/\langle\sigma\rangle$ is a K3 surface, Called the Kummer surface of A

Pf Step 1 X smooth.

π étale outsider the $E_i \Rightarrow$ suff to check smoothness at $\pi(q)$, $q \in E_i$.



get local coordinates (x, y) on A in nbhd of p_i s.t. $\tau^* x = -x$, $\tau^* y = -y$.

set $x' := \varepsilon^* x$
$\quad y' := \varepsilon^* y$

Suppose x' & $t = y'/x'$ local Coordinates on $\hat{A}$ near q

Now $\left| \begin{array}{l} \sigma^* x' = -x' \\ \sigma^* t = (-y')/(-x') = t \end{array} \right.$

$\Downarrow$

t and $u = (x')^2$ form a system of local coo

on X near $\pi(q)$

$\Downarrow$

X Smooth.

Step 2 : Compute canonical divisor of X.

$A = \mathbb{C}^2/\Gamma$ has a holomorphic 2 form $\omega = dx \wedge dy$
abelian surface

nowhere zero

$\Downarrow$

$$\begin{tikzcd}
\hat{A} \arrow[r, "\varepsilon"] \arrow[d, "\pi"'] & A \\
X &
\end{tikzcd}$$

$\tau^*\omega = \omega$

$\Downarrow$

2-form $\varepsilon^*\omega$ is σ-invariant.

$\Downarrow$

$\varepsilon^*\omega = \pi^*\partial$ for some meromorphic 2-form ∂ on X

& $\mathrm{div}(\partial)$ Concentrated on the E_i. let $q \in E_i$.

$$\begin{aligned}
\varepsilon^*\omega = dx' \wedge dy' &= dx' \wedge d(tx') \\
&= x'\, dx' \wedge dt \\
&= \tfrac{1}{2}\, du \wedge dt
\end{aligned}$$

$\Rightarrow$ ∂ holomorphic & non-zero at q.

$\Rightarrow$ $\mathrm{div}(\partial) = 0$ & hence $K_X \underset{lin}{\sim} 0$

Step 3

if X has a non-zero holomorphic 1-form η then $\hat{A}$ has a σ-invariant 1-form.

$$\varepsilon^* : H^0(A, \Omega^1_A) \xrightarrow{\sim} H^0(\hat{A}, \Omega^1_{\hat{A}})$$

$\Rightarrow$ A has a τ-invariant 1-form ξ

($H^0(\Omega^1_A)$ has basis $\{dx, dy\}$)

$\Rightarrow$ $q(X) = 0$ and hence X K3.

$\square$