Canonical bundle formula for elliptic fibrations

Over field $k=\bar{k}$, char $k=0$ (due to Enriques, Kodaira)

THEOREM (CBF)

$f: S \longrightarrow B$ a relatively minimal elliptic fibration

with multiple fibres $m_1F_1, m_2F_2, \cdots, m_kF_k$

then
$$\omega_S = f^*(\omega_B \otimes (R^1f_*\mathcal{O}_S)^\vee) \otimes \mathcal{O}_S\left(\sum_{i=1}^k (m_i-1)F_i\right)$$
$$= f^*(f_*\omega_S) \otimes \mathcal{O}_S\left(\sum_{i=1}^k (m_i-1)F_i\right)$$

pf. denote by F_x the fibre of f over $x \in B$

by F the general fibre of f

Suppose $x_1, \cdots, x_n$ are general points of B, consider the standard exact sequence (after a twisting)

$$0 \longrightarrow \omega_S \longrightarrow \omega_S \otimes \mathcal{O}_S\left(\sum_{i=1}^n F_{x_i}\right) \longrightarrow \bigoplus_{i=1}^n \mathcal{O}_{F_{x_i}} \longrightarrow 0$$

taking cohomology, we obtain a l.e.s. of cohomology groups

$$0 \to \underset{\dim = p_g}{H^0(\omega_S)} \longrightarrow H^0\left(\omega_S \otimes \mathcal{O}_S\left(\sum_i F_{x_i}\right)\right) \to \underset{\dim = n}{H^0\left(\bigoplus_i \mathcal{O}_{F_{x_i}}\right)} \longrightarrow \underset{\dim = q}{H^1(\omega_S)}$$

We infer that

$$h^0\left(\omega_S \otimes \mathcal{O}_S\left(\sum_{i=1}^n F_{x_i}\right)\right) \geq n + p_g - q > 0,$$
for $n \gg 0$

For $n \gg 0$, let $D \in \left|K_S + \sum_{i=1}^n F_{x_i}\right|$, then

$$D.F = K_S F + \sum_i F_{x_i} F = K_S F = 0 \quad (D \text{ vertical})$$

$\Longrightarrow$ We can write

$$\mathcal{O}_S(K_S) = f^*L \otimes \mathcal{O}_S(C)$$

With L a line bundle on base curve B ["fibre part"]

C a curve contained in a union of fibres of f but not containing any fibre. ["fibre components part"]

key point!

For $\forall$ Connected ~~irreducible~~ Component C_0 of C, let F_0 be a fibre of f containing C_0

Claim: F_0 is a multiple fibre (Say $F_0 = mF_0'$ where m is multiplicity of F_0)

C_0 is a rational submultiple of F_0. (i.e. $C_0 = aF_0'$ with $0<a<m$, hence $C_0 = \frac{a}{m}F_0$)

Indeed, let $C_0, C_1, \cdots, C_\ell$ be all connected components of C

then $C = \sum_{i=0}^{\ell} C_i$. By Zariski lemma, $C_i^2 \leq 0$ for each i

$0 = K_S^2 = C^2 = \sum_{i=0}^{\ell} C_i^2$ $\Longrightarrow C_i^2 = 0$ for each i

Say for C_0.

$C_0^2 = 0 \Rightarrow C_0 = r F_0$ for some $r \in \mathbb{Q}_{>0}$

By the choice of C, C_0 is not the whole fibre F_0

$C_0 \subset F_0$

C_0 (effective) curve

$\Rightarrow 0 < r < 1$

F_0 multiple fibre $= m F_0'$

$r = \frac{a}{m}$ with $0 < a < m$

Therefore,

$$\omega_S = f^* L \otimes \mathcal{O}_S\left(\sum_{i=0}^{\ell} a_i F_i'\right)$$

with the $m_i F_i'$ multiple fibres of f

$0 \leq a_i < m_i$

By adjunction formula,

$$\omega_{F_i'} \cong \mathcal{O}_S(K_S + F_i') \otimes \mathcal{O}_{F_i'} \cong \mathcal{O}_S((a_i+1)F_i') \otimes \mathcal{O}_{F_i'}$$

$$\omega_{F_i'} \cong \mathcal{O}_{F_i'}$$

Since for any fibre F_i, $\mathcal{O}_{F_i}(F_i)$ is trivial.

$\Rightarrow \mathcal{O}_S(F_i') \otimes \mathcal{O}_{F_i'}$ is torsion of order m_i, since $m_i F_i'$ is a fibre.

$\Rightarrow a_i + 1 = m_i$ for each i

i.e. $a_i = (m_i - 1)$

$$\Rightarrow \omega_S = f^* L \otimes \mathcal{O}_S\left(\sum (m_i - 1) F_i\right)$$

where $m_i F_i$ multiple fibres of f.

Remark that

$$L = L \otimes f_* \mathcal{O}_S\left(\sum (m_i - 1) F_i\right) \overset{\text{P.F.}}{\cong} f_* \omega_S$$

$$(R^1 f_* \mathcal{O}_S)^\vee \cong \underbrace{f_* \omega_{S/B}}_{\text{Hodge bundle}} \overset{\text{relative duality}}{\cong} (R^1 f_* \mathcal{O}_S)^\vee \otimes \omega_B$$

[For $\forall$ fibration $f: S \to B$, $\forall$ locally free $\mathcal{O}_S$-sheaf $\mathcal{F}$, one has

$$f_*(\mathcal{F}^\vee \otimes \omega_{S/B}) \xrightarrow{\cong} (R^1 f_* \mathcal{F})^\vee]$$

By the well-known formula (obtained via Leray spectral sequence)

$$\chi(\mathcal{O}_S) = \deg f_* \omega_{S/B} + (g-1)(b-1)$$

$$\Rightarrow \deg (R^1 f_* \mathcal{O}_S)^\vee = \deg f_* \omega_{S/B} = \chi(\mathcal{O}_S)$$

$$\deg L = \chi(\mathcal{O}_S) + 2b - 2$$

$$= \chi(\mathcal{O}_S) - 2\chi(\mathcal{O}_B)$$

$\square$

Remark. By canonical bundle formula,

$$K_S \equiv_{\text{num}} \left(\chi(\mathcal{O}_S) - 2\chi(\mathcal{O}_B) + \sum_i \frac{m_i - 1}{m_i}\right) F$$

Kodaira's table of singular fibres

Setting: $f: S \to B$ an elliptic fibration & f relatively minimal

assume $F_0 = f^{-1}(b_0)$ is a singular fibre of f over $b_0 \in B$

Case 1 (F_0 is irreducible)

By adjunction formula, $2p_a(F_0) - 2 = K_S F_0 + F_0^2 \Rightarrow K_S F_0 = 0$

Consider the normalization sequence of F_0 ($\nu: \underset{\text{normalization}}{\tilde{F}_0 \to F_0}$)

$$0 \to \mathcal{O}_{F_0} \to \nu_* \mathcal{O}_{\tilde{F}_0} \to \underset{\substack{\uparrow \\ \text{skyscraper sheaf}}}{\delta} \to 0$$

Taking cohomology

$$0 \to H^0(\delta) \to H^1(\mathcal{O}_{F_0}) \to H^1(\mathcal{O}_{\tilde{F}_0}) \to 0$$

$p_a(F_0) = 1 \Rightarrow h^1(\mathcal{O}_{F_0}) = 1 \Rightarrow h^1(\mathcal{O}_{\tilde{F}_0}) = 0$ or 1

If $h^1(\mathcal{O}_{\tilde{F}_0}) = 1$, then $h^0(\delta) = 0 \Rightarrow F_0$ smooth

If $h^1(\mathcal{O}_{\tilde{F}_0}) = 0$, then $h^0(\delta) = 1 \Rightarrow \overset{\cong \mathbb{P}^1}{\tilde{F}_0}$ smooth rational curve

& $\mathrm{mult}_p(F_0) = 2$

$\swarrow$ P the only singular point of F_0

F_0 is a rational curve with a node [Type I_1 ∞]

or a rational curve with a cusp [Type II]

Case 2 (F_0 reducible, but not multiple)

Claim: $\forall$ irreducible component C_i of $F_0 = \sum n_i C_i$ is a (-2)-curve.

We have proved this before.

$$0 = K_S . F_0 = \sum n_i \underset{\substack{\shortparallel \\ 0}}{K_S C_i} = \sum n_i (-C_i^2 + 2g(C_i) - 2)$$

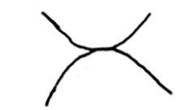
$$\left.\begin{array}{l} C_i^2 \leq -1 \ (\text{Zariski lemma}) \\ C_i^2 = -1,\ p_a(C_i) = 0 \ \text{impossible} \end{array}\right\} \Rightarrow \begin{array}{l} C_i^2 = -2 \\ p_a(C_i) = 0 \end{array}$$

For two different components C_i, C_j

Zariski lemma $\Rightarrow$ $(C_i + C_j)^2 = C_i^2 + 2C_iC_j + C_j^2 \leq 0$

i.e. $\boxed{0 \leq C_i C_j \leq 2}$

If $C_iC_j = 2$, Zariski lemma $\Rightarrow$ $F_0 = C_i + C_j$

type III 

If $C_iC_j < 2$, then any two vertices are joined by at most one edge.

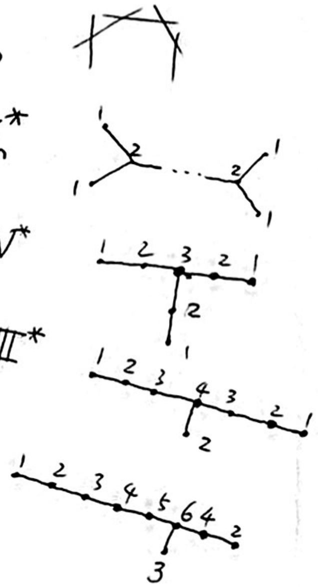
We can consider the associated quadratic form Q_Γ

Zariski lemma $\Rightarrow$ $Q_\Gamma \geq 0$ positive semidefinite

$\Downarrow$

Only possibilities graphs are

$$\left\{\begin{array}{l} \tilde{A}_n \longrightarrow I_n \\ \tilde{D}_{n+4} \longrightarrow I_n^* \\ \tilde{E}_6 \longrightarrow IV^* \\ \tilde{E}_7 \longrightarrow III^* \\ \tilde{E}_8 \longrightarrow \end{array}\right.$$



Case 3) F_0 is a multiple fibre.

$F_0 = m F_0'$

Zariski lemma as before $\Rightarrow$ F_0' is one of the types described before

$\mathcal{O}_S(F_0)$. $\mathcal{O}_{F_0}(F_0)$ both torsion in line bundles

$\Downarrow$

F_0 is of the one of types mI_0, mI_1 & mI_b $\Leftarrow$ F_0' not simply connected.

$\square$

Surfaces with kodaira dimension 1

lemma 1

Let S non-ruled minimal surface, then

(1) if $K_S^2 > 0$, then $\exists\, n_0 \in \mathbb{Z}_{>0}$ s.t. $\phi_{|nK|}$ maps S birationally onto its image for all $n \geq n_0$

(2) if $K_S^2 = 0$ and $P_r = h^0(rK_S) \geq 2$, write $rK \sim \overset{\text{fixed part of } |rK|}{Z} + \underset{\text{moving part}}{M}$

then $KZ = KM = Z^2 = ZM = M^2 = 0$.

Pf. (1) let H be a hyperplane section of S, by Riemann-Roch

$$h^0(nK-H) + h^0(H+(1-n)K) \geq \chi(nK-H) = \chi(\mathcal{O}_S) + \underbrace{\frac{1}{2}(nK-H)((n-1)K-H)}_{\to +\infty \ (\text{as } n \to +\infty)}$$

S non-ruled $\Rightarrow HK_S > 0 \Rightarrow (H+(1-n)K)H < 0$ for $n \gg 0$

$\Downarrow$

$\exists\, n_0$ s.t. $h^0(nK-H) \geq 1$ for all $n \geq n_0$ $\Leftarrow$ $h^0(H+(1-n)K) = 0$ for $n \gg 0$

let $D \in |nK-H|$, then $|nK| = |H+D|$ separates points of $S-D$ (i.e. for two distinct closed pts $P, Q \in S-D$ $\exists\, E \in |H+D|$ s.t. $P \in \operatorname{Supp} E$ & $Q \notin \operatorname{Supp} E$)

& $|H+D|$ also separates tangents to points of $S-D$.

(i.e. given a closed point $p \in S-D$ & a tangent vector at p $v \in T_p(S-D) = (\mathfrak{m}_p/\mathfrak{m}_p^2)^\vee \Rightarrow \exists\, E \in |H+D|$ s.t. $p \in \operatorname{Supp} E$, $v \notin T_pE$)

$\Rightarrow \phi_{|nK|}\big|_{S-D}$ is an embedding

(2)

$$\left.\begin{array}{l} 0 = rK^2 = KZ + KM \\ KZ \geq 0 \\ KM \geq 0 \end{array}\right\} \Rightarrow KZ = KM = 0 \quad (K_S \text{ nef})$$

$$\left.\begin{array}{l} M \text{ movable} \Rightarrow ZM \geq 0, M^2 \geq 0 \\ 0 = rKM = MZ + M^2 \end{array}\right\} \Rightarrow MZ = M^2 = 0$$

$$Z^2 = (rK - M)^2 = 0$$

Prop. If S minimal surface with $\chi = 1$, then

(1) $K_S^2 = 0$

(2) $\exists$ a smooth curve B and a surjective morphism $f: S \longrightarrow B$ whose generic fibre is an elliptic curve.

S called an elliptic su

Pf. By lemma 1 (1), $K_S^2 \leq 0$.

Fact: S minimal & $K_S^2 < 0 \Rightarrow S$ ruled (will be proved later)

$\Rightarrow K_S^2 = 0$.

$\kappa(S)=1 \Rightarrow \exists\, r \in \mathbb{Z}_{>0}$ s.t. $P_r = h^0(rK_S) \geq 2$

Write $rK_S \underset{\text{lin}}{\sim} Z + M$, where Z fixed part of $|rK_S|$

M movable part of $|rK_S|$

by above lemma, $M^2 = KM = 0 \Rightarrow |M|$ bpf & the image of

$$\varphi_{|M|} : S \longrightarrow \mathbb{P}^{P_r-1}$$

has dimension one, say $C = \operatorname{Im}\varphi_{|M|}$

Consider the Stein factorization of $\varphi_{|M|}$

$$\varphi_{|M|} : S \xrightarrow{f} B \longrightarrow C \subset \mathbb{P}^{P_r-1}$$

then $f: S \to B$ is a fibration. Let F be a fibre of f

$$\left.\begin{array}{l} \left.\begin{array}{l} M \text{ is a sum of fibres of } f \\ KM = 0 \\ K \text{ nef} \end{array}\right\} \Rightarrow KF = 0 \\ F \text{ a fibre of } f \Rightarrow F^2 = 0 \end{array}\right\} \Rightarrow p_a(F) = 1$$

By generic smoothness, the generic fibre of f is a smooth elliptic curve.

□

By previous prop. all surfaces with $\kappa=1$ are elliptic.

The converse is not true, but we have

Prop Let S minimal elliptic surface with an elliptic fibration

$$f: S \longrightarrow B, \quad \underset{\text{fibre}}{F_b} \longmapsto b$$

then (1) $K_S^2 = 0$

(2) S is either ruled over an elliptic base

or surface with $\kappa = 0$

or surface with $\kappa = 1$

(3) If $\kappa(S) = 1$, then $\exists$ integer $d > 1$ s.t.

$dK \sim \sum_i n_i F_{b_i}$ where $n_i \in \mathbb{Z}_{>0}$ & $b_i \in B$

For $r \gg 0$, $|rdK|$ bpf and thus defines a morphism $\varphi: S \longrightarrow \mathbb{P}^N$ which factors through f:

$$\varphi : S \xrightarrow{f} B \xhookrightarrow{j} \mathbb{P}^N$$

Pf. If S ruled over a curve C, then the elliptic fibres F_b ($S \overset{\text{bir}}{\simeq} C \times \mathbb{P}^1$)

must be mapped surjectively onto C

$\Rightarrow$ C is either rational or elliptic

$\Rightarrow$ $K_S^2 = 8(1-g(C)) \geq 0$

By lemma 1, for non-ruled minimal surface S

$$K_S^2 > 0 \Leftrightarrow \kappa(S) = 2 \quad (\text{eq. } 0 \le \kappa(S) \le 1 \Leftrightarrow K_S^2 = 0)$$

$\Rightarrow$ for all minimal elliptic surfaces, $K^2 \geq 0$

Suppose that $\exists$ some integer n s.t. $|nK| \neq \phi$ (eq. $h^0(nK) \geq 1$)

let $D \in |nK|$

f elliptic $\Rightarrow KF_b = 0 \Rightarrow DF_b = 0$ for $\forall b$
($p_a(F_b) = 1$)
for each $b \in B$

$\Downarrow$

D f-vertical, that is

Components of $D \subseteq$ fibres of f.

$\Downarrow$ Zariski lemma

$$\left.\begin{array}{l} D^2 \leq 0 \\ \parallel \\ 0 \leq n^2K^2 \end{array}\right\} \Rightarrow \begin{array}{l} D^2 = 0 \\ \parallel \\ K^2 \end{array}$$

&

$D = \sum r_i F_{b_i}$

for some $r_i \in \mathbb{Q}_{\geq 0}$

Now let X be a minimal elliptic surface with $K_X^2 > 0$

Claim : Such a surface X does not exist.

Indeed, since suppose $\exists\, n$ s.t. $|nK| \neq \phi$, we would have $K^2 = 0$.

Thus $|nK| = \phi$ for all $0 \neq n \in \mathbb{Z}$.

$$h^0(nK) + h^0((1-n)K) \geq \chi(nK) = \chi(\mathcal{O}_S) + \frac{1}{2} n(n-1)K^2$$

$\downarrow$

$+\infty$

(as $n \to +\infty$)

let S be a minimal elliptic surface, then $K_S^2 = 0$. [(1) holds]

$p_a(F_b) = 1 \Rightarrow KF_b = 0 \Rightarrow K$ is f-vertical

$\Downarrow$

the maps $\phi_{|nK|}$ contract the fibres F_b

$\Downarrow$

$\dim \operatorname{Im} \phi_{|nK|} \leq 1 \Rightarrow \kappa = -\infty, 0$ or 1.

[(2) holds]

Case $\kappa = 1$.

Choose an integer n s.t. $P_n \geq 1$. let $D \in |nK|$, then $D = \sum r_i F_{b_i}$ with $r_i \in \mathbb{Q}_{\geq 0}$, write $r_i = \frac{n_i}{m}$ ($m, n_i \in \mathbb{Z}_{>0}$) & put $d = mn$

then

$$dK \sim mD \sim \sum n_i F_{b_i} = f^*A \quad \text{where } A = \sum n_i [b_i]$$

For $r \gg 0$,

$$|rA| \text{ bpf \& very ample} \Rightarrow \overset{|rA|}{} \text{ defines an embedding } j : B \xhookrightarrow{|rA|} \mathbb{P}^N$$

$$|rdK| = f^*|rA| \text{ bpf \& defines a morphism}$$

$$\varphi = \varphi_{|rdK|} : S \xrightarrow{f} B \xhookrightarrow{j} \mathbb{P}^N$$

[(3) holds]

$\square$

<u>Example</u> (surfaces with $\kappa = 1$)

B : smooth curves

$|D|$: bpf linear system on B

$$\begin{tikzcd}
S \arrow[r, hook] \arrow[d, "p"'] & \overset{X}{\overset{\shortparallel}{B \times \mathbb{P}^2}} \arrow[dl, "p"'] \arrow[dr, "q"] & \\
|D| \quad B & & \mathbb{P}^2
\end{tikzcd}$$

by Bertini's theorem,

a general member $S \in |\mathcal{O}_B(D) \boxtimes \mathcal{O}_{\mathbb{P}^2}(3)|$ is smooth

the restriction $p : S \longrightarrow B$ is a fibration by plane cubics (i.e. elliptic curves)

By adjunction formula $\qquad g = \frac{(d-1)(d-2)}{2}$

$$\omega_S \cong \omega_X \otimes \mathcal{O}_X(S)|_S = p^*\omega_B \otimes q^*\mathcal{O}_{\mathbb{P}^2}(-3) \otimes p^*\mathcal{O}_B(D) \otimes q^*\mathcal{O}_{\mathbb{P}^2}(3)|_S$$
$$= p^*\mathcal{O}_B(K_B + D)|_S$$

$\Rightarrow K_S \sim p^*(K_B + D)$

When $\deg D > 2 \ \cancel{2g(B)} \ \cancel{2} \ \cancel{2g(B)}$,

$$\Rightarrow |n(K_B+D)| \text{ defines an embedding } j : B \hookrightarrow \mathbb{P}^N$$

$|K_B + D|$ ~~very~~ very ample $\Rightarrow |nK_S|$ bpf

$\Rightarrow |nK_S|$ defines a morphism factoring through p

$$\varphi_{|nK_S|} : S \xrightarrow{p} B \xhookrightarrow{j} \mathbb{P}^N$$

In particular, $\kappa(S) = 1$