Smooth Morphisms

Recall $f: X \to Y$ smooth of rel. dim n if ① f flat ② $\begin{array}{ccc} X & \xrightarrow{f} & Y \\ \cup & & \cup \\ X' & \longrightarrow & Y' \\ \text{irred comp} & & \text{irred comp.} \end{array}$ then $\dim X' = \dim Y' + n$ ③ $\dim_{k(x)} \Omega_{X/Y} \otimes k(x) = n$. $\forall x \in X$

lemma 1 let $f: X \to Y$ be a dominant morphism of nonsingular irreducible varieties over an alg closed field k of char 0.

then $\exists$ a nonempty open subset $V \subseteq X$ s.t.

$$f: \cancel{f^{-1}(V)} V \longrightarrow Y \text{ is a smooth morphism.}$$

That is for $\forall y \in Y$, the fiber $f^{-1}(y)$ is geometrically smooth (non-empty) of pure dimension $\dim X - \dim Y$.

Pf.

- For $\forall$ variety X, the function field $K(X)$ is finitely generated$/k$.
- $f: X \to Y$ dominant $\Rightarrow$ $\exists$ k-alg homo. $K(Y) \to K(X)$
- $k = \bar{k}$, any finitely generated field extension$/k$ is separably generated

$\Rightarrow$ $K(X)$ is a separably generated field extension of $K(Y)$

$\Rightarrow$ $\dim_{K(X)} \Omega_{K(X)/K(Y)} = \text{tr.deg. } K(X)/K(Y)$

$\Rightarrow$ $\Omega_{X/Y}$ is free of rank $= \dim X - \dim Y$ at the generic point of X

$\Rightarrow$ $\Omega_{X/Y}$ is locally free of rank $(\dim X - \dim Y)$ on some nonempty open subset $V \subseteq X$

X, Y nonsingular $\Rightarrow$ $f: V \longrightarrow Y$ smooth

lemma 2 $f: X \to Y$ morphism of schemes of finite type$/k = \bar{k}$, char $k = 0$

For each $r \geq 0$, let $T_{f,x}: T_{X,x} \to T_{Y,f(x)}$ be the tangent map & $X_r := \{ \text{closed points } x \in X \mid \text{rank } T_{f,x} \leq r \}$,

then $\dim \overline{f(X_r)} \leq r$.

Pf let V be an irreducible component of $\overline{f(X_r)}$ &

let U be an irreducible component of X_r s.t. $\begin{cases} f(U) \subseteq V \\ \& \\ \overline{f(U)} = V. \end{cases}$

Consider U and V with reduced induced scheme structures & morphism $f: U \longrightarrow V$

By lemma 1, $\exists$ nonempty open subset $U' \subset U$ s.t.

$$f|_{U'}: U' \longrightarrow V \text{ smooth}$$

Now consider $x \in U' \cap X_r$, and the following commutative diagram

$$\begin{tikzcd} T_{U',x} \arrow[r, "\text{surjective}"', "\text{since } U' \to V \text{ smooth}"] \arrow[d, hook] \arrow[dr, phantom, "\circlearrowright"] & T_{V,f(x)} \arrow[d, hook] \\ T_{X,x} \arrow[r, "T_{f,x}"'] & T_{Y,f(x)} \end{tikzcd} \Rightarrow \dim T_{V,f(x)} \leq r$$

rank $T_{f,x} \leq r$ since $x \in X_r$

$$\Downarrow$$

$$\dim V \leq r.$$

THEOREM (Generic Smoothness)

$f : X \longrightarrow Y$ a morphism of Var. $/ k = \bar{k}$, char $k = 0$

assume X is nonsingular.

then $\exists$ non-empty open subset $V \subseteq Y$ s.t.

$$f : f^{-1}(V) \longrightarrow Y \text{ is smooth.}$$

Pf. By the proof of lemma 1,

$\exists$ an open dense subset $U \subseteq Y$ which is non-singular.

So WMA Y nonsingular. let $r = \dim Y$

then $\dim \overline{f(X_{r-1})} \leq r-1$

Removing $\overline{f(X_{r-1})}$ from Y, WMA $\operatorname{rank} T_f \geq r$ for $\forall$ closed pts of X.

Y nonsingular of dim $r \Rightarrow T_f$ surjective for $\forall$ closed pts of X

$\Downarrow$

f smooth.

[Rmk If f not dominant, then $V \subseteq Y - \overline{f(X)}$ & $f^{-1}(V)$ will be empty.]

Morphisms to a curve

Let $f : S \longrightarrow B$ a surjective morphism from a smooth projective surface S to a nonsingular projective curve B.

then
- f is projective and hence proper
- f is flat (by "miracle flatness")

$\Downarrow$

the arithmetic genus of $F_b = f^{-1}(b)$ is independent of $b \in B$

Consider the Stein factorization of f

$$S \xrightarrow{g} C \xrightarrow{h} B, \qquad C = \operatorname{Spec}_B(f_* \mathcal{O}_S)$$

where $g : S \longrightarrow C$ with $g_* \mathcal{O}_S = \mathcal{O}_C \xRightarrow{\text{Zariski's connected theorem}}$ g has connected fibres

$h : C \longrightarrow B$ finite morphism

S normal $\Rightarrow C$ normal, i.e. C also smooth

[If ϕ rational function on C which integral over $\mathcal{O}_C$ then $g^*\phi = \phi \circ g$ integral over $\mathcal{O}_S \Rightarrow \phi \circ g \in \mathcal{O}_S \Rightarrow \phi \in g_* \mathcal{O}_S = \mathcal{O}_C$]

$g_* \mathcal{O}_S = \mathcal{O}_C \Rightarrow$ all but finitely many fibres of g are integral curves. (irreducible & reduced)

Zariski lemma

$f: S \to C$ fibration over a smooth projective curve C

F a fibre of f over $x \in C$

Clearly two projections

$$\begin{array}{ccc} F \times C & \xrightarrow{pr_2} & C \\ {\scriptstyle pr_1}\downarrow & & \\ F & & \end{array}$$

are fibrations

Called trivial fibrations

lemma 1 | the normal sheaf of F, $\mathcal{O}_F(F)$ is trivial, that is $\mathcal{O}_F(F) \cong \mathcal{O}_F$

Pf. let $x = f(F)$, and D be a divisor on C with $D \underset{lin}{\sim} x$ & $x \notin \mathrm{Supp}\, D$

then $\mathcal{O}_F(F) \cong \mathcal{O}_F(f^*D) \cong \mathcal{O}_F$

lemma 2 (Zariski lemma)

let $F = \sum_{i=1}^{l} n_i F_i$ be a fibre of a fibration $f: S \to C$

where F_i irreducible components of F, $n_i > 0$, then

(1) $FF_i = 0$ for all i

(2) For any divisor $D = \sum_{i=1}^{l} m_i F_i$, $m_i \in \mathbb{Z}$, one have

$D^2 \leqslant 0$, and $D^2 = 0 \Leftrightarrow D$ is a rational multiple of F.

($\exists\, r \in \mathbb{Q}$, s.t. $D = rF$)

Pf. (1) let F' be another fibre of f, then $F_i \cap \mathrm{Supp}\, F' = \phi$.

and thus $FF_i = F'F_i = 0$

(2) $\dfrac{\text{Suppose that } D^2 > 0.}{\text{by (1), } FD = 0} \xRightarrow[F \text{ not numerically trivial}]{\text{Hodge index thm}} F^2 < 0$ $\lightning$

Hence $D^2 \leqslant 0$

Now assume that $D^2 = 0$.

let $r = \min\{r \in \mathbb{Q} \mid D + rF \geqslant 0 \text{ as } \mathbb{Q}\text{-divisors}\}$

then clearly there are at least 1 components of F not contained in $D + rF$.
(by the choice of r)

If D not a multiple of F, then $D + rF \neq 0$ (not trivial divisor

$\Rightarrow \exists$ an irreducible component Γ of F, such that

$$\begin{cases} \Gamma \not\subset D + rF \\ \Gamma \cap \mathrm{Supp}(D + rF) \neq \phi \end{cases}$$

$\Rightarrow \Gamma(D + rF) = \Gamma D > 0$

$\Rightarrow (mD + \Gamma)^2 = 2m\Gamma D + \Gamma^2 > 0$ for $m \gg 0$ $\lightning$ (Since $mD + \Gamma = \sum k_i F_i$ we must have $(mD+\Gamma)^2 \leqslant 0$

Application of Zariski lemma

Estimation on Picard number $\rho(S)$

F a fibre of a fibration $f : S \longrightarrow C$

$\ell(F) := \#\{\text{irreducible components of } F\}$

Theorem (lower bound on $\rho(S)$)

$$\rho(S) \geq 2 + \sum_{F \text{ reducible fibres}} \left(\ell(F) - 1\right)$$

Pf. let $F_1, \cdots, F_m$ be all irreducible fibres of f.

For each i,

$F_{i1}, F_{i2}, \cdots, F_{i\ell(F_i)}$ all irred. components of F_i

let H be an ample divisor of S & F be a general fibre.

then H, F and all F_{ij} $(j < \ell(F_i))$ are numerically independent.

Indeed, Suppose $D = aH + bF + \sum_{i=1}^{m} \sum_{j=1}^{\ell(F_i)-1} c_{ij} F_{ij} \underset{num}{\equiv} 0$.

for certain integers $a, b, c_{ij} \in \mathbb{Z}$.

then $\underset{\underset{aHF}{\parallel}}{DF} = 0 \Rightarrow a = 0$

$$0 = D^2 = \left(bF + \sum_i \sum_j c_{ij} F_{ij}\right)^2 = \sum_{i=1}^{m} \left(\sum_{j=1}^{\ell(F_i)-1} c_{ij} F_{ij}\right)^2$$

By Zariski lemma, $\Rightarrow \left(\sum_{j=1}^{\ell(F_i)-1} c_{ij} F_{ij}\right)^2 \leq 0$

$\Downarrow$

$$\left(\sum_{j=1}^{\ell(F_i)-1} c_{ij} F_{ij}\right)^2 = 0$$

$c_{ij} = 0 \Leftarrow \sum_{j=1}^{\ell(F_i)-1} c_{ij} F_{ij} = r_i F_i$ for some $r_i \in \mathbb{Q}$

$\Downarrow$

$D = bF$

$$\left.\begin{array}{l} bHF = HD = 0 \quad (\text{since } D \underset{num}{\equiv} 0) \\ HF > 0 \end{array}\right\} \Rightarrow b = 0$$

$\square$

Elliptic fibrations

Def Let S be a smooth projective surface. A fibration (pencil) on S is a surjective morphism $f: S \longrightarrow B$ to a smooth projective curve B, and all fibres of f are connected.

(S also called a pencil of curves)

A general fibre is irreducible & smooth, its genus g called the genus of the fibration / the genus of pencil.

The singular fibres are said to be degenerate.

B called the base curve, its genus denoted by $b := g(B)$.

The fibration f is relatively minimal if for $\forall\, b \in B$, the fibre $F_b = f^{-1}(b)$ does not contain any (-1)-curve among its components.

Def A fibration $f: S \longrightarrow B$ is elliptic if the general fibre of f is a smooth elliptic curve.

Rmk For a relatively minimal fibration $f: S \longrightarrow B$ (with $f_* \mathcal{O}_S = \mathcal{O}_B$) if $p_a(F_b) = 1$ for some $b \in B$.

Over $k = \bar{k}$, $\mathrm{Char}\, k = 0$,

f is generically smooth $\Rightarrow$ almost all fibres of f (except finitely many closed fibres) are smooth elliptic curves

$\Downarrow$

f is an elliptic fibration.

Set-up: $f: S \longrightarrow B$ an elliptic fibration

f relatively minimal

$F_b = \sum_{i=1}^{\ell} m_i F_i$ is a fibre of f over $b \in B$

Def Let $n \in \mathbb{Z}_{>0}$ be the largest positive integer such that $\frac{1}{n} F_b$ is an integral divisor, then F_b is called a multiple fibre & n is the multiplicity of F_b.

Clearly $n = \gcd(m_1, \cdots, m_\ell)$

Observation. $F_b . F_i = K_S . F_i = 0$ for $\forall\, 1 \le i \le \ell$.

Pf. If F is a fibre of f, distinct from F_b, then $F \underset{num}{\equiv} F_b$

• $F_b . F_i = F . F_i = 0$ for $\forall\, i$, since $\mathrm{Supp}(F) \cap F_i = \phi$.

· By genus formula,

$$2\underset{\substack{\| \\ 1}}{p_a(F_b)} - 2 = \underset{\substack{\| \\ 0}}{F_b^2} + K_S F_b \Rightarrow \underset{\substack{\| \\ \sum_{i=1}^{\ell} m_i K_S F_i}}{K_S F_b} = 0.$$

if $\ell = 1$, $m_1 \geq 1$, then $K_S . F_1 = 0$

if $\ell \geq 2$, then by ~~Grauert's criterion~~ Zariski's lemma, $(F_i)^2 < 0$ for $\forall 1 \leq i \leq \ell$.

$$\left.\begin{array}{l} 2p_a(F_i) - 2 = F_i^2 + K_S F_i \\ K_S F_i < 0 \text{ for some } i \end{array}\right\} \Rightarrow \underset{F_i \text{ smooth rational curve}}{p_a(F_i) = 0 \ \& \ F_i^2 = -1}$$

↯ (f relatively minimal)

$$\Rightarrow \left.\begin{array}{l} K_S F_i \geq 0 \text{ for } \forall 1 \leq i \leq \ell. \\ \sum_{i=1}^{\ell} m_i K_S F_i = 0 \end{array}\right\} \Rightarrow K_S F_i = 0 \text{ for } \forall 1 \leq i \leq \ell.$$

Observation | If F_b is a reducible fibre of f ($\ell \geq 2$), then each irreducible component F_i of F_b is a (-2)-curve.

$$\left.\begin{array}{l} p_a(F_i) = 0 \\ K_S F_i = 0 \end{array}\right\} \Rightarrow \left.\begin{array}{l} 2\underset{\substack{\| \\ 0}}{p_a(F_i)} - 2 = F_i^2 + \underset{\substack{\| \\ 0}}{K_S F_i} \Rightarrow F_i^2 = -2 \\ p_a(F_i) = 0 \Rightarrow F_i \text{ smooth rational curve} \end{array}\right\} F_i \ (-2)\text{-curve.}$$

Def. S minimal surface

$D = \sum_{i=1}^{\ell} n_i E_i \geq 0$ an effective divisor on S

then D is a <u>divisor/curve of canonical type</u> if

$$K_S E_i = D \cdot E_i = 0 \quad \text{for } \forall 1 \leq i \leq \ell.$$

If moreover D is connected & $\gcd(n_1, \cdots, n_\ell) = 1$

then D called an <u>indecomposable</u> curve of canonical type.

Theorem let $D = \sum_{i=1}^{\ell} n_i E_i \geq 0$ be an indecomposable curve of canonical type on a minimal surface S & let L be an invertible $\mathcal{O}_S$-module.

If $\deg(\underset{\substack{\| \\ L \otimes \mathcal{O}_{E_i}}}{L|_{E_i}}) = 0$ for $\forall 1 \leq i \leq \ell$, then

$$H^0(D, L) \neq 0 \iff \begin{cases} L \cong \mathcal{O}_D \\ H^0(D, \mathcal{O}_D) \cong \mathbb{C} \end{cases}$$

Corollary If D is an indecomposable curve of canonical type, then $\omega_D \cong \mathcal{O}_D$, here ω_D is the dualizing sheaf of D.

Pf By Serre duality (for embedded curves),

$$\dim H^1(\omega_D) = \dim H^0(\mathcal{O}_D) \underset{\substack{\uparrow \\ D \text{ connected}}}{=} 1$$

Consider the exact sequence

$$0 \to K_S \to K_S + D \to (K_S+D)|_D \to 0$$

$(K_S+D)|_D \cong \omega_D$ ← dualizing sheaf

$$\Rightarrow \chi(\omega_D) = \chi(K_S+D) - \chi(K_S)$$
$$\overset{R.R.}{=} \frac{1}{2}(K_S+D)D \underset{\substack{\uparrow \\ \text{by definition of indecomposable curve of canonical type}}}{=} 0$$

$\Rightarrow \dim H^0(\omega_D) = 1$

$\deg(\omega_D|_{E_i}) = (K_S+D).E_i = 0$ for $\forall i$.

$\overset{\text{Thm}}{\Longrightarrow} \omega_D \cong \mathcal{O}_D$. □

Cor If $D = \sum_{i=1}^{\ell} n_i E_i$ is an indecomposable curve of canonical type & D' is an effective divisor on surface S s.t. $D'.E_i = 0$ for each $1 \leq i \leq l$.

then $D' = nD + D''$ where $n \geq 0$, $D'' \geq 0$ effective divisor with $\operatorname{Supp} D'' \cap \operatorname{Supp} D = \phi$.

Pf. let n be the non-negative integer defined by the conditions

$$\begin{cases} D' - nD \geq 0 \\ D' - (n+1)D \text{ not effective} \end{cases}$$

Put $D'' := D' - nD$ & $L := \mathcal{O}_D(D'')$

Consider the exact sequence

$$0 \to \mathcal{O}_S(D''-D) \to \mathcal{O}_S(D'') \to \mathcal{O}_D(D'') \to 0$$

$\mathcal{O}_D(D'') = L$

- $D'' \geq 0 \Rightarrow \exists$ nonzero section $s \in H^0(S, \mathcal{O}_S(D''))$ s.t. $\operatorname{div}(s) = D''$.
- $D'' - D = D' - (n+1)D$ not effective $\Rightarrow s \notin H^0(D''-D) \Rightarrow \bar{s} \in H^0(D,L)$, $\bar{s} \neq 0$

$$0 \to H^0(D''-D) \to H^0(D'') \to H^0(D, L)$$
$$s \longmapsto \bar{s}$$

- $\deg(L|_{E_i}) = D''.E_i = D'E_i - nDE_i = 0$ for all i

by Theorem, $L \cong \mathcal{O}_D$ & $H^0(D,L) \cong \mathbb{C} \Rightarrow s(x) \neq 0$ for $\forall x \in D \Rightarrow \operatorname{Supp} D'' \cap \operatorname{Supp} D = \phi$.

THEOREM

S minimal surface with $K_S^2 = 0$ & K_S ~~minimal~~ nef

(i.e. $K_S C \geq 0$ for all curves $C \subset S$)

If S contains an indecomposable curve D of canonical type then $\exists$ an elliptic fibration $f: S \to B$ on S.

More precisely, $\exists\, n > 0$ s.t. the Stein factorization of the morph. $\varphi_{|nD|}: S \longrightarrow \mathbb{P}^{h^0(nD)-1}$ is a pencil of curves of canonical type, that is an elliptic fibration.

<u>Pf</u> [Case $p_g = 0$.]

$$0 \to \mathcal{O}_S(-D) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_D \to 0$$

twisted by $\mathcal{O}_S(nK_S + nD)$ $(n \geq 0)$

$$0 \to \mathcal{O}_S(nK + (n-1)D) \longrightarrow \mathcal{O}_S(nK+nD) \longrightarrow \underbrace{\mathcal{O}_S(nK+nD)|_D}_{\substack{\cong \\ \mathcal{O}_D}} \to 0$$

$\cong$ ← Corollary $\omega_D \cong \mathcal{O}_D$

<u>Claim</u> : $H^2(nK + (n-1)D) = 0$ for all $n \geq 2$

and thus $H^2(nK + nD) = 0$ for $\forall\, n \geq 2$.

Granting this claim for the moment,

D indecomposable curve of canonical type $\Rightarrow H^1(\mathcal{O}_D) = H^0(\omega_D) = H^0(\mathcal{O}_D) \neq 0$

$$\Rightarrow \underset{\text{exact}}{\to H^1(nK+nD) \to H^1(\mathcal{O}_D) \to} \underbrace{H^2(nK + (n-1)D)}_{=0}$$

$H^1(nK+nD) \neq 0$

by Riemann-Roch

$D = \sum_{i=1}^{\ell} n_i E_i$

$K E_i = D \cdot E_i = 0$ for each i

$$\chi(nK+nD) = \chi(\mathcal{O}_S) + \frac{1}{2}(nK+nD)((n-1)K+nD)$$
$$= \chi(\mathcal{O}_S)$$
$$= 1 - q(S)$$

by Noether formula

$$12\chi(\mathcal{O}_S) = K_S^2 + e(S)$$

$$\underbrace{12(1-q)}_{\boxed{p_g=0}} \overset{\boxed{K^2=0}}{=} 2 - 2b_1 + b_2 = 2 - 4q + b_2 \Rightarrow 10 - 8q = b_2 \overset{p_g = 0}{=} h^{1,1} \geq 0$$

$\Downarrow$

$q \leq 1$

$$\Rightarrow \chi(nK+nD) = 0 \text{ or } 1 \quad (\text{for } n \geq 2)$$

$$\| $$

$$h^0(nK+nD) - \underbrace{h^1(nK+nD)}_{\neq 0}$$

$\Rightarrow h^0(nK+nD) \neq 0$ for $n \geq 2$.

$\Rightarrow$ for $n \geq 2$, $\exists$ effective divisor $D_n \in |nK+nD|$.

<u>Observation 1</u> $D_n \neq 0$.

(Otherwise, $nK+nD \underset{\text{lin.}}{\sim} 0$ (i.e. $nK \sim -nD$)

let C be an irreducible curve on S s.t. $D.C > 0$

then $K_S . C = \frac{1}{n}(nK.C) = -D.C < 0$ ↯ (K_S nef))

Observation 2 D_n is of canonical type.

Indeed, say $D = \sum_{i=1}^{\ell} n_i E_i$, then $kE_i = D\cdot E_i = 0$ &

$$D_n \cdot E_i = (nk+nD).E_i = 0 \quad \text{for all } i$$

by Corollary. $D_n = mD + \underbrace{\sum k_j F_j}_{F\geq 0}$ $\begin{cases} m\geq 0 \text{ integer} \\ k_j > 0 \\ F_j \text{ irred. curve with } F\cap \operatorname{Supp} D = \phi. \end{cases}$

$kF = \sum k_j kF_j \geq 0$, since k nef

$$\left.\begin{array}{l} kF \underset{kD=0}{=} k(F+mD) = kD_n = k(nk+nD) = 0 \end{array}\right\} \Rightarrow kF_j = 0 \text{ for } \forall j.$$

$$D_n\cdot F_j = (nk+nD)F_j = nkF_j + n\underbrace{DF_j}_{F_j\cap \operatorname{Supp} D = \phi} = 0$$

By definition, D_n is of canonical type.

Observation 3 Cannot have $D_n = a_n D$ for integers $a_n \geq 0$ & $\forall n\geq 2$.

(Otherwise, $D_n \in |nk+nD| \Rightarrow nk \underset{\text{lin.}}{\sim} b_n D$ for $b_n \in \mathbb{Z}$ & $n\geq 2$

In particular $k = 3k-2k \sim (b_3-b_2)D = bD$ for some $b\in\mathbb{Z}$

if $b<0$, take an irred. curve C s.t. $DC>0$, then $kC = bDC < 0$ $\lightning$

if $b\geq 0$, $|k_S| = |bD| \neq \phi$ $\lightning$)

End of proof Construct a pencil from two distinct indcomposable curves of canonical type

We have seen that $\exists \geq 1$ indecomposable curve of canonical type D' on S, which is disjoint from D

$$0\to k_S \to k_S + D \to \omega_D \to 0 \quad \text{exact}, \qquad \omega_D \cong \mathcal{O}_D$$

$$0\to k_S \to k_S + D' \to \omega_{D'} \to 0 \quad \text{exact}, \qquad \omega_{D'} \cong \mathcal{O}_{D'}$$

twisted by $\mathcal{O}_S(D)$ (resp. $\mathcal{O}_S(D')$)

$$0\to \mathcal{O}_S(k_S+D)\to \mathcal{O}_S(k_S+2D)\to \mathcal{O}_D\to 0$$
$$0\to \mathcal{O}_S(k_S+D')\to \mathcal{O}_S(k_S+2D')\to \mathcal{O}_{D'}\to 0 \quad \text{exact}$$

$\Rightarrow$
$$0\to \underbrace{\mathcal{O}_S(k_S+D)\otimes\mathcal{O}_S(k_S+D')}_{\mathcal{O}_S(2k_S+D+D')} \to \underbrace{\mathcal{O}_S(k_S+2D)\otimes\mathcal{O}_S(k_S+2D')}_{\mathcal{O}_S(2k_S+2D+2D')} \to \mathcal{O}_D\oplus\mathcal{O}_{D'} \to 0 \quad \text{exact}$$

$(t\otimes t') \longmapsto (t,t')$

Similar argument as before, we have

$$H^2(2k+D+D') = 0 \text{ and thus } H^2(2k+2D+2D') = 0$$

$\Rightarrow \to H^1(2k+2D+2D') \to \underbrace{H^1(\mathcal{O}_D\oplus\mathcal{O}_{D'})}_{\text{2-dim'l}} \to 0 \Rightarrow h^1(2k+2D+2D')\geq 2$

by Riemann-Roch,

$$\underset{\substack{\| \\ h^0 - h^1}}{\chi(2K+2D+2D')} = \chi(\mathcal{O}_S) + \frac{1}{2}(2K+2D+2D')(K+2D+2D')$$
$$= \chi(\mathcal{O}_S)$$
$$= 1-q = 0 \text{ or } 1$$

$\Rightarrow h^0(2K+2D+2D') \geq 2$

let $\Delta \in |2K+2D+2D'|$, consider the complete linear system $|\Delta|$

Remark that

① $\Delta > 0$ (strictly positive divisor)

② $\Delta^2 = (2K+2D+2D')^2 = 0$

③ $\dim|\Delta| \geq 1$

④ Δ is of canonical type, since $K^2=0$ & D, D' of canonical type

⑤ $|\Delta|$ is composed with a pencil & $\varphi_{|\Delta|}$ is a morphism :

let C be a fixed part of $|\Delta|$, then $|\Delta - C|$ no fixed components

$(\Delta-C)^2 = \Delta^2 - 2\Delta C + C^2 = C^2 \underset{\uparrow}{\leq} 0$

$\#\{\text{base points of } |\Delta-C|\} \leq (\Delta-C)^2$ $\left.\begin{matrix} \\ \text{Zariski lemma} \end{matrix}\right\} \Rightarrow |\Delta - C|$ free (no base points)

$\Rightarrow$ the rational map defined by $|\Delta|$

$$\varphi_{|\Delta|} : S \longrightarrow \phi_{|\Delta|}(S) := B \subset |\Delta|$$

is a morphism.

- $\dim|\Delta| \geq 1 \Rightarrow B$ cannot be a point
- If B was a surface, then $\Delta - C = \varphi^* H$ for a hyperplane section H of B

$\Rightarrow (\Delta-C)^2 = \underset{\uparrow}{[k(S):k(B)]}\, H^2 > 0$ $\lightning$

generic degree of the generically finite morphism $\varphi: S \to B$

$\Rightarrow |\Delta|$ is composed with a pencil (i.e. B is a curve) & $\varphi_{|\Delta|}$ is a morphism.

⑥ Consider the Stein factorization of $\varphi_{|\Delta|} : S \xrightarrow{f} B' \longrightarrow B$
then $f: S \to B$ is an elliptic fibration.

$D(\Delta - C) \geq 0$, since $D \geq 0$, $\Delta - C \geq 0$ & D. each its comp. $E_i = 0$
$D\Delta = D(2K+2D+2D') = 0$
$DC \geq 0$ since $D \geq 0$, $C \geq 0$ & D. each its comp. $E_i = 0$

$D(\Delta - C) = 0$

$\left.\begin{array}{l} D \text{ Connected} \\ D.(\Delta - C) = 0 \end{array}\right\} \Rightarrow$ D contained in a fibre of $\varphi_{|\Delta|}$

$\&$

$D^2 = 0$

$\Downarrow$ Zariski lemma

$D = rF_{b'}$ for some $r \in \mathbb{Q}$

fibre $F_{b'}$ of f over $b' \in B'$

$F_{b'} = \frac{1}{r} D$ with $\frac{1}{r} \in \mathbb{Z}_{>0} \Leftarrow D = \sum_{i=1}^{\ell} n_i E_i$ $\quad \gcd(n_1, n_2, \cdots, n_\ell) = 1$

$\uparrow$

positive integral multiple of D

$p_a(F_{b'}) = p_a(D) = h^0(\omega_D) = h^0(\mathcal{O}_D) = 1 \Rightarrow f$ is an elliptic fibration

[Case $p_g > 0$]

Suffices to show $\dim H^0(\Delta) \geq 2$ for some divisor Δ of Canonical type

Specifically, we shall prove that $\exists n > 0$ s.t.

$$\dim H^0(nD) \geq 2.$$

Take a non-zero section $s \in H^0(nD)$. Consider the s.e.s.

$$0 \to \mathcal{O}_S \xrightarrow{\cdot s} \mathcal{O}_S(nD) \longrightarrow \underbrace{\mathcal{O}_S(nD)/\mathcal{O}_S}_{= F_n} \to 0$$

taking cohomology

$$0 \to H^0(\mathcal{O}_S) \to H^0(nD) \longrightarrow H^0(F_n) \longrightarrow H^1(\mathcal{O}_S)$$

Suffices to show that $h^0(F_n) \to +\infty$ as $n \to +\infty$.

Put $L = F_1 = \mathcal{O}_S(D)/\mathcal{O}_S$ it is an invertible sheaf on D

$$0 \to F_{n-1} \to F_n \longrightarrow L^{\otimes n} \longrightarrow 0 \quad \text{exact}$$

$\Rightarrow$ $h^0(F_n) \geq h^0(F_{n-1})$ i.e. $h^0(F_n)$ nondescreasing function in n.

By Riemann-Roch,

D canonical type

$$\chi(nD) = \chi(\mathcal{O}_S) + \frac{1}{2} nD(nD - K_S) = \chi(\mathcal{O}_S)$$

$\Downarrow$ additivity of Euler-Poincaré characteristic

$\chi(F_n) = 0$ for all $n \geq 1$

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(nD) \longrightarrow \underbrace{\mathcal{O}_S(nD)/\mathcal{O}_S}_{\parallel \; F_n} \to 0$$

$$\to H^1(F_n) \to H^2(\mathcal{O}_S) \to H^2(nD) \to 0$$

$$H^2(S, \mathcal{O}_S(nD)) \overset{\text{Serre}}{\cong} H^0(k_S - nD)^\vee = 0 \quad \text{for } n \gg 0$$

[E any divisor on surface S, $D > 0$ (strictly positive div)

then $|E - nD| = \phi$ for $n \gg 0$

Indeed, let H be an ample divisor, then $HD > 0$

& $(E - nD)H < 0$ for $n \gg 0$

if $|E - nD| \neq \phi$, then $H(E - nD) \geq 0$

$$\left.\begin{array}{l} p_g = h^2(\mathcal{O}_S) > 0 \\ h^2(nD) = 0 \text{ for } n \gg 0 \end{array}\right\} \Rightarrow h^1(F_n) \neq 0 \quad \text{for } n \gg 0.$$

$$\chi(F_n) = 0 \text{ for } n > 0 \Rightarrow h^0(F_n) = h^1(F_n) \quad \text{for } n > 0.$$

$\Rightarrow$

$$h^0(F_n) = h^1(F_n) \neq 0 \quad \text{for } n \gg 0.$$

<u>Assume that</u> the sequence of integers $\{h^0(F_n)\}$ is bounded above, and let n be the largest index s.t. $h^0(F_{n-1}) < h^0(F_n)$.

$$0 \to F_{n-1} \to F_n \longrightarrow L^{\otimes n} \longrightarrow 0$$

$$0 \to H^0(F_{n-1}) \to H^0(F_n) \to H^0(L^{\otimes n}) \to H^1(F_{n-1})$$

$$s \longmapsto t \neq 0$$

$\Rightarrow$ $H^0(L^{\otimes n}) \neq 0$ & $\exists$ nonzero section $t \in H^0(L^{\otimes n})$ s.t. it is the image of a section $s \in H^0(F_n)$

$$\left.\begin{array}{l} \overset{\sum n_i E_i}{\overset{\parallel}{D}} \text{ indecomposable curve of canonical type} \\ \deg(L^{\otimes n}|_{E_i}) = nDE_i = 0 \text{ for } \forall i \\ H^0(L^{\otimes n}) \neq 0 \end{array}\right\} \Rightarrow L^{\otimes n} \cong \mathcal{O}_D \ \& \ H^0(\mathcal{O}_D) \cong \mathbb{C}$$

$\Downarrow$

$s|_D$ does not vanishing at any point of D

$\mathrm{Supp}(F_n) = D \Rightarrow$ s generates F_n as $\mathcal{O}_S$-module at $\forall$ points of S

$\Downarrow$

s defines a surjection of $\mathcal{O}_S$-modules

$$\mathcal{O}_S \twoheadrightarrow F_n$$

with kernel $\mathcal{O}_S(-nD)$

$$\Longrightarrow \mathcal{O}_S/\mathcal{O}_S(-nD) \cong F_n \cong \mathcal{O}_S(nD)/\mathcal{O}_S$$

~~Tensoring with F_n~~ For $m \geq 1$, $F_{nm} \cong \mathcal{O}_S(nmD)/\mathcal{O}_S$

$$F_{nm}/F_{n(m-1)} \cong \frac{\mathcal{O}_S(nmD)}{\mathcal{O}_S(n(m-1)D)} \cong \mathcal{O}_S/\mathcal{O}_S(-nD) \cong F_n$$

$$\begin{array}{ccccccccc}
 & & & & 0 & & 0 & & \\
 & & & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_S & \to & \mathcal{O}_S(n(m-1)D) & \to & F_{n(m-1)} & \to & 0 \\
 & & \| & & \downarrow & & \downarrow & & \\
0 & \to & \mathcal{O}_S & \to & \mathcal{O}_S(nmD) & \to & F_{nm} & \to & 0 \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & F_n & = & F_n & & \\
 & & & & \downarrow & & \downarrow & & \\
 & & & & 0 & & 0 & &
\end{array}$$

taking cohomology

$$\begin{array}{ccccccc}
H^1(F_{n(m-1)}) & \xrightarrow{\alpha} & H^1(F_{nm}) & \xrightarrow{\beta} & H^1(F_n) & \longrightarrow & 0 \\
\downarrow & & \downarrow & & & & \\
H^2(\mathcal{O}_S) & = & H^2(\mathcal{O}_S) & & & & \\
\downarrow & & \downarrow & & & & \\
H^2(n(m-1)D) & \xrightarrow{\sim} & H^2(nmD) & = 0 & \text{for } m \gg 0 & &
\end{array}$$

$$\left.\begin{array}{l} p_g > 0 \Leftrightarrow \\ H^1(F_{n(m-1)}) \twoheadrightarrow H^2(\mathcal{O}_S) \end{array}\right\} \Rightarrow \alpha : H^1(F_{n(m-1)}) \to H^1(F_{nm})$$

nonzero map

$$\Downarrow$$

$$\dim H^1(F_{nm}) = \dim H^1(F_n) + \dim(\operatorname{Im}\alpha) > h^1(F_n) \quad ↯$$

(a contradiction to the choice of n.) $\square$

<u>proof of Claim</u> : $H^2(nk + (n-1)D) = 0$ for all $n \geq 2$.

$$H^2(nk + (n-1)D) \overset{\text{Serre}}{\cong} H^0((1-n)k + (1-n)D)^\vee$$

Suffices to show $H^0(-m(k+D)) = 0$ for all $m \geq 1$.

(eq. $|-m(k+D)| = \phi$)

By way of contradiction, assume $\exists$ effective div. $\Delta \in |-m(k+D)|$

then

Case 1) $\Delta = 0$.

$$m(k+D) \sim 0 \Rightarrow mk \sim -mD.$$

let C be an irreducible curve s.t. $DC > 0$

then $K_S C = \frac{1}{m}(mk.C) = -\frac{1}{m}(mD.C) = -D.C < 0$ ↯

Case 2) $\Delta > 0$ (strictly positive divisor)

$$\left.\begin{array}{l} \text{let } C \text{ be an irreducible curve such that } \Delta C > 0. \\ \Delta \cdot C = -m(K_S + D).C \leq 0 \end{array}\right\} ↯$$

K_S nef $\Rightarrow K_S C \geq 0$

$DC \geq 0$ since D is of canonical type. $\square$

THEOREM

let S be a minimal surface with $K_S^2=0$ & K_S nef,

then either $2K_S \sim 0$

or S Contains at least one indecomposable curve of canonical type.

Pf. <u>Case $|2K_S| \neq \phi$.</u>

let $D \in |2K_S|$, then either $D \sim 0$ (i.e. $2K_S \sim 0$)

or $D = \sum_{i=1}^{l} n_i E_i > 0$

if $D = \sum_{i=1}^{l} n_i E_i > 0$, then

$$\left.\begin{array}{l} KD = 2K^2 = 0 \\ \quad \| \\ \sum_{i=1}^{l} n_i K E_i \\ K \text{ nef} \end{array}\right\} \Rightarrow KE_i = 0 \text{ for each } i.$$

$DE_i = 2KE_i = 0$ for each i $\Longrightarrow$ D is of Canonical type

In particular, S Contains at lease 1 indecomposable curve of Canonical type.

<u>Case $|2K_S| = \phi$.</u> $h^0(2K) = 0$

by Riemann-Roch,

$$h^0(2K) + h^0(-K) \geq \chi(2K) = \chi(\mathcal{O}_S) + K^2 = \chi(\mathcal{O}_S) = 1-q$$

$\Rightarrow$ ~~dim |2K| +~~ $\dim |-K| \geq -q$

If $q=0$, then $|-K| \neq \phi \Rightarrow \exists\, D \in |-K|$

if $D=0$, then $2K \underset{\text{lin.}}{\sim} 0$ $\lightning$ (Since $|2K| = \phi$)

if $D>0$ (strictly positive eff. div.), let H be a hyperplane section of S

then $HD > 0 \Rightarrow HK = -HD < 0$ $\lightning$ (K nef)

Hence $q \geq 1$.

By Noether formula,

$$\underset{\underset{12(1-q)}{\|}}{12\chi} = K^2 + e(S) = 2 - 4q + b_2 \Rightarrow b_2 = 10 - 8q \geq 0$$

$$\Downarrow$$

$$q \leq 1$$

$$\Downarrow$$

$$\boxed{q=1}$$

Consider the Albanese morphism of S

$$a_S : S \longrightarrow E = \mathrm{Alb}\, S$$

$\uparrow$ elliptic curve

which is a fibration over a smooth elliptic curve E.

let $F_b = a_S^{-1}(b)$ be a fibre over $b \in E$.

If $p_a(F_b)=0$, then $F_b^2=0$

$$2p_a(F_b)-2 = F_b^2 + KF_b \Rightarrow KF_b = -2 \ \lightning$$

(K nef)

If $p_a(F_b)=1$, then $F_b^2=0$ & $KF_b=0 \Rightarrow F_b$ is a curve of canonical type

$\Downarrow$

S contains an indecomposable curve of canonical type

If $p_a(F_b)\geq 2$.

$$2p_a(F_b)-2 = F_b^2+KF_b = KF_b \geq 2$$

For each closed point $b' \in E\setminus\{b\}$, consider the s.e.s.

$$0\to \mathcal{O}_S(2K+F_{b'}-F_b)\to \mathcal{O}_S(2K+F_{b'})\to \mathcal{O}_{F_b}(2K+F_{b'}|_{F_b})\to 0$$

Claim: $|F_b - F_{b'} - K| = \phi$ (eq. $h^0(F_b-F_{b'}-K)=h^2(2K+F_{b'}-F_b)=0$)

(Indeed, otherwise suppose $0\leq D\in|F_b-F_{b'}-K|$, then

$K \underset{\text{lin.}}{\sim} F_b - F_{b'} - D$

$KF_b = (F_b-F_{b'}-D)F_b = -DF_b \overset{\text{since } D\geq 0 \ \&\ F_b^2=0}{\leq} 0 \ \lightning$ ($KF_b\geq 2$))

again by Riemann-Roch

$$\underset{h^0-h^1}{\chi}(2K+F_{b'}-F_b) = \chi(\mathcal{O}_S)+\frac{1}{2}(2K+F_{b'}-F_b)(K+F_{b'}-F_b)$$
$$= \chi(\mathcal{O}_S)$$
$$= 0$$

$P_2(S)=0 \Rightarrow p_g=0$
$q(S)=1$

$\Rightarrow$ for $\forall\, b'\in E\setminus\{b\}$, we have one of the following two possibilities

Case (1) $h^0(2K+F_{b'}-F_b)\geq 1$ (i.e. $|2K+F_{b'}-F_b|\neq\phi$)

Case (2) $h^0(2K+F_{b'}-F_b)=0$ (hence $h^i(2K+F_{b'}-F_b)=0$ for $\forall i$)
(i.e. $|2K+F_{b'}-F_b|=\phi$)

Claim: the case (2) cannot occur for all $b'\in E\setminus\{b\}$.

Assume that $|2K+F_{b'}-F_b|=\phi$ for all $b'\in E$. Then

$$\mathcal{O}_{F_b}(2K+F_{b'}|_{F_b}) \cong \mathcal{O}_{F_b}(2K|_{F_b}) \qquad \boxed{2p_a(F_b)-2 = KF_b+F_b^2 = KF_b}$$

$$h^0(2K|_{F_b}) \geq \chi(2K|_{F_b}) \overset{R.R.}{=} \deg(2K|_{F_b})+1-p_a(F_b)$$
$$= 2K.F_b+1-p_a(F_b)$$
$$= 3(p_a(F_b)-1)\geq 3 \ (\text{since } p_a(F_b)\geq 2)$$

Fix a nonzero section $t\in H^0(\mathcal{O}_{F_b}(2K|_{F_b}))$ & Put $\Delta = \operatorname{div}_{F_b}(t)$

$$0 \to H^0(2k+F_{b'}-F_b) \to H^0(2k+F_{b'}) \overset{\sim}{\to} H^0(F_b,\ 2k|_{F_b}) \to H^1(2k+F_{b'}-F_b)$$

$$\begin{array}{ccccc} \| & & & & \| \\ 0 & & t_{b'} \longmapsto t & & 0 \end{array}$$

for $\forall\ b' \in E\setminus\{b\}$, $\Rightarrow \exists!\ t_{b'} \in H^0(2k+F_{b'})$ s.t. it is a lifting of t

let $D_{b'} := \mathrm{div}_S(t_{b'})$

$\rightsquigarrow$ we get an algebraic family of effective divisors $\{D_{b'}\}_{b'\in E\setminus\{b\}}$

$$\begin{array}{cc} D_{b'} & |D_{b'}| \\ \downarrow & \downarrow \\ b' \in E\setminus\{b\} & \end{array}$$

such that $D_{b'}|_{F_b} = \Delta$ for all b'

Moreover, if $b', b'' \in E\setminus\{b\}$ & $b' \neq b''$, then $D_{b'} \underset{lin}{\not\sim} D_{b''}$.

(Otherwise, $D_{b'} \underset{lin}{\sim} D_{b''} \Rightarrow 2k+F_{b'} \underset{lin}{\sim} 2k+F_{b''} \Rightarrow F_{b'} \sim F_{b''}$

$\Downarrow$

$b' \sim b''$ on elliptic curve E $\lightning$)

In particular, $D_{b'} \neq D_{b''}$. then $S = \overline{\bigcup_{b'\neq b} D_{b'}}$.

as $b' \rightsquigarrow b$, (D_b) must contain F_b among its components.

$\Rightarrow D_b = F_b + \widetilde{D_b}$ for some $\widetilde{D_b} \geq 0$.

Since $D_b \in |2k+F_b| \Rightarrow \widetilde{D_b} \in |2k|$ $\lightning$ $(|2k|=\phi)$

Summary $|2k+F_{b'}-F_b| \neq \phi$ for at least one $b' \in E$

If $D \in |2k+F_{b'}-F_b|$, say $D = \sum_{i=1}^{l} n_i E_i$

$$\begin{array}{l} kD = k(2k+F_{b'}-F_b) = 2k^2 = 0 \\ \| \\ \sum_{i=1}^{l} n_i kE_i \end{array} \Rightarrow kE_i = 0 \text{ for each } i$$

$DE_i = (2k+F_{b'}-F_b)E_i = 2kE_i = 0 \Rightarrow DE_i = 0$ for each i

then D is of canonical type

$\square$

Cor

S minimal surface with $K_S^2 = 0$ & K_S nef

then either $2K_S \sim 0$

or $\exists$ an elliptic fibration $f: S \to B$ on S.