Minimal models of ruled surfaces

Setting :

C smooth projective curve

E locally free $\mathcal{O}_C$-module of rank 2

$S = \mathbb{P}(E)$ projective bundle associated with E

$$\begin{tikzcd} S \arrow[r, "\pi"] & C \\ F_b \arrow[u, phantom, "\cup"] \arrow[r, mapsto] & b \arrow[u, phantom, "\in"] \\ x \arrow[u, phantom, "\in"] \end{tikzcd}$$

$\pi : S \longrightarrow C$ the canonical projection

$\varepsilon : \widetilde{S} \longrightarrow S$ the blow-up of S at $x \in F_b \subset S$

F', (-1)-curve E'_x ; F_b, x

$$\begin{tikzcd} \widetilde{S} \arrow[r, "\varepsilon"] \arrow[d, "\sigma"'] & S \arrow[d, "\pi"] \\ S' & C \end{tikzcd}$$

$\sigma(E'_x)$, x'

Let $F' \subset \widetilde{S}$ be the proper transform of F_b on S, then

$$\left.\begin{array}{l} \varepsilon^* F_b = F' + E'_x \Rightarrow (\varepsilon^* F_b)^2 = E_x'^2 + 2E'_xF' + F'^2 \\ (\varepsilon^* F_b)^2 = F_b^2 = 0 \end{array}\right\} \Rightarrow F'^2 = -1$$

$F' \longrightarrow F_b$ & $p_a(F_b) = 0 \Rightarrow p_a(F') = 0 \Rightarrow F' \cong \mathbb{P}^1$ & $F'^2 = -1$

by Castelnuovo's contractibility criterion $\Rightarrow \exists$ a blow-down $\widetilde{S} \xrightarrow[= \mathrm{Cont}_{F'}]{\sigma} S'$

which contracts F' to a smooth point x' of S'.

$$\begin{tikzcd} F' \arrow[dd, bend right, "\mathrm{Cont}_{F'}"'] & & x \in F_b \\ \widetilde{S} \arrow[r, "\varepsilon = \mathrm{Bl}_x S"] \arrow[d, "\mathrm{Cont}_{F'} = \sigma"'] & S \arrow[d, "\pi"] \\ x' \quad S' & C \quad b \end{tikzcd}$$

F' a component of a fiber of $\pi \circ \varepsilon \Rightarrow \exists$ a morphism $\pi' : S' \longrightarrow C$ such that

$$\begin{tikzcd} \widetilde{S} \arrow[r, "\varepsilon"] \arrow[d, "\sigma"'] \arrow[dr, phantom, "\circlearrowright"] & S \arrow[d, "\pi"] \\ S' \arrow[r, "\pi'"'] & C \end{tikzcd}$$

$\pi' : S' \longrightarrow C$ is a geometrically ruled surface $\Longleftarrow$ all fibers of π' are integral curves of arithmetic genus 0 $(\cong \mathbb{P}^1)$

the birational map $S \overset{\sigma \circ \varepsilon^{-1}}{\dashrightarrow} S'$ ($= \mathrm{elem}_x$) called the elementary transformation of the geometrically ruled surface $\pi : S \longrightarrow C$ with center $x \in S$

$\left\{\begin{array}{l} \text{undefined at } x \in F_b \text{ (fiber)} \\ \text{Contracting all points of } F_b \text{ but } x \text{ to a single point } x' \in S' \\ \text{isom. outside } F_b \end{array}\right.$

THEOREM

let B be a smooth projective curve with $g(B) > 0$, then

(a) If $\pi : S \longrightarrow B$ is a geometrically ruled surface with base curve B, then X is a minimal model.

(b) For $\forall$ minimal model S of the field $k(B)(t)$ $\left\{\begin{array}{l} k(B) \text{ rat'l function field} \\ t \text{ an indepent variable}/k(B) \end{array}\right.$ then $\exists$ a morphism $\pi : S \longrightarrow B$ such that S is a geometrically ruled surface with base B via π.

(c) $\forall$ birational map between two minimal models S, S' of $K(B)(t)$ is a composite of an automorphism of S & a finite number of elementary transformations elem_x.

<u>Pf</u> (a) By contradiction, assume that S contains a (-1)-curve E.

$$\left.\begin{array}{l} E \text{ rational curve} \\ g(B)>0 \end{array}\right\} \Rightarrow E \text{ vertical, that is, contained in a fiber of } \pi$$

Since all the fibers of π are integral curves,

$$E \text{ must coincide with a fiber of } \pi$$
$$\Downarrow$$
$$E^2=0 \quad ↯.$$

Parts (b) & (c) follow from the following lemma.

<u>Lemma</u> E' locally free sheaf of rank 2 on a sm proj curve B with $g(B)>0$

S : surface

$f : S \dashrightarrow \mathbb{P}(E')$ a birational map

then $\exists$ a finite product $g : \mathbb{P}(E') \dashrightarrow \mathbb{P}(E)$ of elementary transformations such that

$$\begin{tikzcd} S \arrow[r, dashed, "f"] \arrow[rr, bend right, "g\circ f"'] & \mathbb{P}(E') \arrow[r, dashed, "g"] & \mathbb{P}(E) \end{tikzcd}$$

a morphism.

<u>Pf</u>. let $g : \mathbb{P}(E') \dashrightarrow \mathbb{P}(E)$ be an arbitrary finite product of elementary transformations &

$$\begin{tikzcd} S \arrow[r, dashed, "f"] \arrow[rr, bend right, "h := g\circ f"'] & \mathbb{P}(E') \arrow[r, dashed, "g"] & \mathbb{P}(E) \end{tikzcd}$$

let $\lambda = \lambda(h)$ be the minimal number of blow-ups of S needed to get a surface X that dominates S &

$$\begin{tikzcd} X \arrow[d, "\pi"', "\text{blowups}"] \arrow[dr, "h' \text{ morphism}"] \arrow[dr, phantom, "\circlearrowleft" near start] & \\ S \arrow[r, dashed, "h"'] & \mathbb{P}(E) \end{tikzcd}$$

Now choose g such that $\lambda = \lambda(h)$ is the smallest possible.

<u>Claim</u>: $\lambda = 0$ & thus $h : S \longrightarrow \mathbb{P}(E)$ is a morphism.

Assume by Contradiction that $\lambda > 0$. Then X contains a (-1)-curve C such that the surface $X' := \mathrm{Cont}_C(X)$ dominates S

$$\begin{tikzcd} C \subseteq X \arrow[d, "\mathrm{cont}_C"] \arrow[dd, bend right, "\pi"'] \arrow[ddr, "h'"] & \\ X' \arrow[d] & \\ S \arrow[r, dashed, "h"'] & \mathbb{P}(E) \end{tikzcd}$$

- If $h'(C)$ was a point in $\mathbb{P}(E)$ then $\exists$ a morphism $X' \longrightarrow \mathbb{P}(E)$ ↯ (to minimality of λ)

Hence $D := h'(C)$ is a curve in $\mathbb{P}(E)$

C smooth rational curve $\Rightarrow$ D is also a rational curve

$$\left.\begin{array}{l} g(B) > 0 \\ \pi : \mathbb{P}(E) \longrightarrow B \end{array}\right\} \Rightarrow$$ D must be π-vertical, that is D is a fiber of $\mathbb{P}(E) \to B$

$$\Downarrow$$

$$D^2 = 0.$$

$$\begin{tikzcd}
C \arrow[rrd, mapsto, bend left] & & \\
X \arrow[d, "cont_C"] \arrow[dd, bend right=60, "\substack{\pi \\ \text{blow-ups}}"'] \arrow[ddr, "h'"] & & D \\
X' \arrow[d] & & \\
S \arrow[r, dashed, "h"] & \mathbb{P}(E) &
\end{tikzcd}$$

If the birational map $(h')^{-1} : \mathbb{P}(E) \dashrightarrow X$ had no fundamental points on D (i.e. "undefined points")

then this morphism h' would be a biregular isomorphism of an open nbhd of C onto an open nbhd of D

but $D^2 = 0$, $C^2 = -1$ $\lightning$.

$\Longrightarrow$ $\exists$ a point $x \in D$ that is a fundamental point of the birat'l map $(h')^{-1} : \mathbb{P}(E) \dashrightarrow X$

$$\begin{tikzcd}
\overset{C}{X} \arrow[d, "cont_C"] \arrow[dd, bend right=60, "\pi"'] \arrow[ddr, "h'"] \arrow[r] & Bl_x(\mathbb{P}(E)) \arrow[d] \arrow[dr] & \\
X' \arrow[d] & & \\
S \arrow[r, dashed, "h"] & \mathbb{P}(E)^{x \in D} \arrow[r, dashed, "elem_x"] & \mathbb{P}(\mathcal{F})
\end{tikzcd}$$

by the universal property of blowing-up,

$\exists$ a morphism $X \longrightarrow Bl_x(\mathbb{P}(E))$ s.t.

$$\begin{tikzcd}
X \arrow[r] \arrow[dr, "h'"'] & Bl_x(\mathbb{P}(E)) \arrow[d] \arrow[dl, phantom, "\circlearrowright"] \\
& \mathbb{P}(E)
\end{tikzcd}$$

recall that $x \in D$ which is a fiber of $\mathbb{P}(E) \to B$

$\Rightarrow$ $\exists$ an elementary transformation of geometrically ruled surfaces $elem_x : \mathbb{P}(E) \dashrightarrow \mathbb{P}(\mathcal{F})$ for some rank 2 v.b. $\mathcal{F}$ on B

[undefined at x, contracting all points (but x) of D to a single point of $\mathbb{P}(\mathcal{F})$]

$\Rightarrow$ $X \xrightarrow{\text{dominates}} elem_x(\mathbb{P}(E))$

$\quad\;\; \overset{\cup}{C} \longmapsto \overset{\in}{pt}$

$\Rightarrow$ $X' \longrightarrow elem_x(\mathbb{P}(E))$ a morphism & hence

$\lambda(elem_x \circ h) < \lambda(h) = \lambda$ $\lightning$ (to the choice of g) $\square$

Rational surfaces.

Recall a surface S is rational if it is birationally equivalent to the projective plane $\mathbb{P}^2$.

Remark. • $\mathbb{P}^2 \overset{bir}{\sim} \mathbb{P}^1 \times \mathbb{P}^1$

$\Rightarrow$ $\forall$ rational surface is a ruled surface S with $q(S)=0$.

• Conversely, if S is a ruled surface with $q(S)=0$,

then $S \overset{bir}{\sim} \mathbb{P}^1 \times B$ for some smooth projective curve.

$$\Rightarrow \underset{\underset{0}{\|}}{q(S)} = \underset{\underset{0}{\|}}{g(\mathbb{P}^1)} + g(B) \Rightarrow g(B)=0.$$

$$B \cong \mathbb{P}^1$$

$\Rightarrow$ S is a rational surface.

Geometrically ruled surfaces / $\mathbb{P}^1$

In this part, we consider the geometrically ruled surface $S \xrightarrow{\pi} \mathbb{P}^1$

then $S \cong \mathbb{P}(E)$ over $\mathbb{P}^1$ for some rank 2 vector bundle E on $\mathbb{P}^1$.

By (the special case of) Grothendieck's theorem,

$E \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b)$ for suitable integers $a \leq b$.

$$\begin{aligned} E &\cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(b) \\ &= \mathcal{O}_{\mathbb{P}^1}(a) \otimes \left(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(b-a)\right) \\ &= L \otimes \left(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)\right) \end{aligned}$$

with L an invertible $\mathcal{O}_{\mathbb{P}^1}$-module

$n = b-a \geq 0$ an integer

then $\mathbb{P}(E) \cong \mathbb{P}\left(\mathcal{O}_{\mathbb{P}^1} \oplus \mathcal{O}_{\mathbb{P}^1}(n)\right) := \mathbb{F}_n$ $(n \geq 0)$

let $f_n : \mathbb{F}_n \longrightarrow \mathbb{P}^1$ be the canonical projection. then we have two sections of f_n :

zero section $\sigma_0 : \mathbb{P}^1 \longrightarrow \mathbb{F}_n$ $\quad \sigma_0(\mathbb{P}^1) := D_0 \subset \mathbb{F}_n$

section at infinity $\sigma_\infty : \mathbb{P}^1 \longrightarrow \mathbb{F}_n$ $\quad \sigma_\infty(\mathbb{P}^1) := D \subset \mathbb{F}_n$.

their conormal bundles in $\mathbb{F}_n$ given by

$$\mathcal{O}_{D_0}(-D_0) = \mathcal{O}_{\mathbb{P}^1}(n)$$

$$\mathcal{O}_{D}(-D) = \mathcal{O}_{\mathbb{P}^1}(-n)$$

$$\Rightarrow D_0^2 = -n \quad \& \quad D^2 = n$$

Recall $\mathrm{Pic}(\mathbb{F}_n) \cong \mathbb{Z}[\text{Section}] \oplus f_n^* \mathrm{Pic}(\mathbb{P}^1)$

$\mathrm{Num}(\mathbb{F}_n) \cong \mathbb{Z}\{\text{section}\} \oplus \mathbb{Z}\{\text{fiber of } f_n\}$

$\Rightarrow$ for $\forall$ divisor Δ on $\mathbb{F}_n$

$$\Delta \underset{num}{\equiv} aD + bF$$

where the pair (a, b) uniquely determined

F a closed fiber of f_n.

Then
$$\Delta^2 = (aD + bF)^2 = a^2 D^2 + 2ab\,DF + b^2 F^2 = a^2 n + 2ab$$

$$\Delta' \underset{num}{\equiv} cD + eF$$

$$\Delta\Delta' = (aD + bF)(cD + eF) = ac\,D^2 + (ae+bc)DF + be\,F^2 = acn + ae + bc$$

$\Rightarrow (-,-) : \mathrm{Num}(\mathbb{F}_n) \times \mathrm{Num}(\mathbb{F}_n) \longrightarrow \mathbb{Z}$

$(a, b) \cdot (c, e) \longmapsto acn + ae + bc$

- If Δ effective divisor, then $\underset{\substack{\| \\ a}}{\Delta F} \geq 0$

(Recall for $\forall$ embedded curve $C \subset S$,

$$p_a(C) = 1 + \frac{1}{2}(K_S C + C^2))$$

$$p_a(\Delta) = 1 + \frac{1}{2}(K\Delta + \Delta^2)$$

$$\left.\begin{aligned} &p_a(D) = p_a(F) = 0 \\ &D^2 = n \\ &F^2 = 0 \end{aligned}\right\} \Rightarrow \begin{aligned} &p_a(D) = 1 + \tfrac{1}{2}(KD + D^2) \\ &p_a(F) = 1 + \tfrac{1}{2}(KF + F^2) \end{aligned}$$

$$\Downarrow$$

$$KD = -2-n$$
$$KF = -2$$

$$\Downarrow$$

$$K\Delta = K(aD + bF) = a(-2-n) - 2b$$

$$\Downarrow$$

$$p_a(\Delta) = 1 + \frac{1}{2}(a^2 n + 2ab - 2a - an - 2b)$$

- On another (zero) section $D_0 \subset \mathbb{F}_n$

$\mathrm{Supp}(D_0) \cap \mathrm{Supp}(D) = \phi \Rightarrow D_0 D = 0$.

If $D_0 \underset{num}{\equiv} aD + bF$, then

$$\left.\begin{aligned} &\underset{\substack{\| \\ 1}}{D_0 F} = \underset{\substack{\| \\ a}}{aD.F + bF^2} \Rightarrow a = 1 \\ &\underset{\substack{\| \\ 0}}{D_0 D} = \underset{\substack{\| \\ n+b}}{aD^2 + bFD} \Rightarrow b = -n \end{aligned}\right\} \Rightarrow D_0 \underset{num}{\equiv} D - nF$$

- If Δ an effective divisor that doesn't contain D_0 as a component,

  & $\Delta \underset{hum}{\equiv} aD + bF$

  then $b \geq 0$

Indeed,

$$\begin{array}{c} \Delta D_0 \geq 0 \\ \| \\ aDD_0 + bFD_0 \\ \| \\ b \end{array}$$

<u>Conclusion</u>. | if Δ is an integral curve on $\mathbb{F}_n$, then we cannot have $\Delta^2 < 0$ unless $\begin{cases} n > 0 \\ \Delta = D_0 \end{cases}$

In other words, for $n \geq 0$, then on $\mathbb{F}_n$, D_0 is the unique integral curve with negative self-intersection number.

Indeed, if $\Delta \neq D_0$, then $\Delta \underset{hum}{\equiv} aD + bF$

Δ effective $\Rightarrow$ $a \geq 0$ & $b \geq 0$

$$\Downarrow$$

$$\Delta^2 = a^2 n + 2ab \geq 0$$

if $\Delta = D_0$, then $\Delta^2 = D_0^2 = -n$

Note that $p_a(D_0) = 0$, then we have

<u>Lemma</u>

- $\mathbb{F}_n$ are minimal models of $\mathbb{P}^2$ for all $n \geq 0$, $n \neq 1$.
- if $n > 0$, then the only integral curve with negative self-intersection number on $\mathbb{F}_n$ is D_0, the zero section of $\mathbb{F}_n \to \mathbb{P}^1$
- surface $\mathbb{F}_1$ contains exactly one (-1)-curve D_0 (zero section)

  & $\mathrm{Cont}_{D_0}(\mathbb{F}_1) \underset{\text{biregular isom.}}{\cong} \mathbb{P}^2$ i.e. $\mathbb{F}_1 \longrightarrow \mathbb{P}^2$ is a blow-up at a single point.
- Surface $\mathbb{F}_0 \underset{\text{biregular isom.}}{\cong} \mathbb{P}^1 \times \mathbb{P}^1$.

(Denote by $\mathrm{refl} : \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1 \longrightarrow \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ the reflection morph)

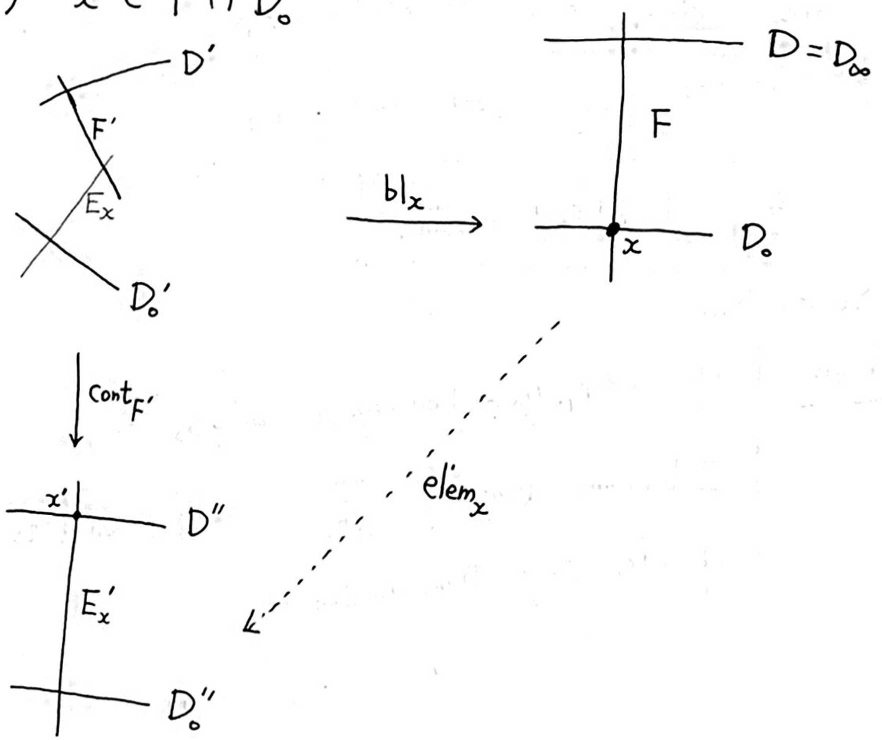
$$(x, y) \longmapsto (y, x)$$

Birational maps between these $\mathbb{F}_n$

① let $x \in \mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$. then $\mathrm{elem}_x(\mathbb{F}_0) \cong \mathbb{F}_1$

② For $n \geq 1$, let $x \in \mathbb{F}_n$ and F be the fiber of $f_n : \mathbb{F}_n \longrightarrow \mathbb{P}^1$ through x.

(Case 2.1) $x \in F \cap D_0$



$$bl_x^* D_0 = D_0' + E_x \Rightarrow \underbrace{(bl_x^* D_0)^2}_{= D_0^2 = -n} = D_0'^2 + 2\underbrace{D_0' E_x}_{} + \underbrace{E_x^2}_{} $$

$$(bl_x^* D_0)^2 = D_0^2 = -n, \qquad D_0'^2 + 2D_0'E_x + E_x^2 = (D_0')^2 + 1$$

$$\Rightarrow (D_0')^2 = -n-1$$

$$(D')^2 = D^2 = n$$

$$(\mathrm{cont}_{F'}^* D'' = D' + F')$$

$$\underbrace{(\mathrm{cont}_{F'}^* D'')^2}_{= D''^2} = (D' + F')^2 = \underbrace{(D')^2 + 2D'F' + F'^2}_{= n+1}$$

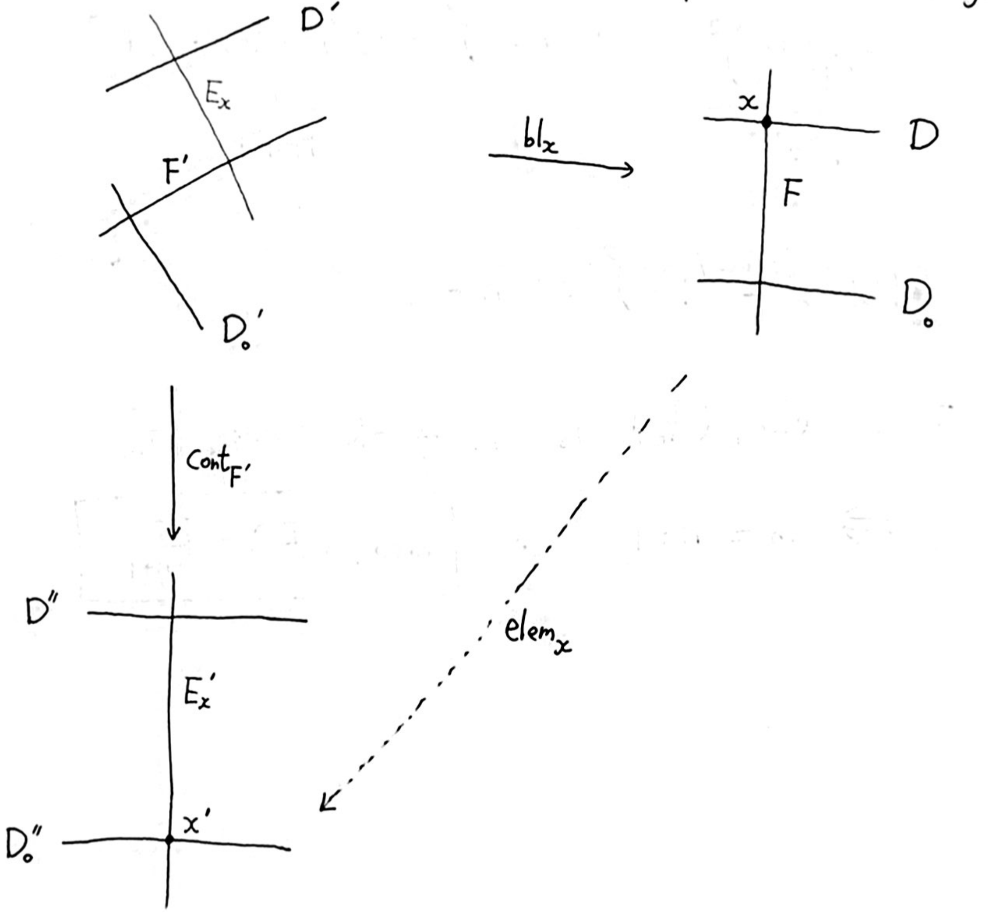
$$(D_0'')^2 = (D_0')^2 = -(n+1)$$

Since $\mathrm{elem}_x(\mathbb{F}_n)$ is one of the surfaces $\mathbb{F}_m$

$\Rightarrow m = n+1$ i.e. $\boxed{\mathrm{elem}_x(\mathbb{F}_n) = \mathbb{F}_{n+1}}$.

(Case 2.2) $\quad x \notin F \cap D_0$.

WLOG, WMA $x \in D$ because we can change the point at infinity on $\mathbb{P}^2$ while keeping the point $0 \in \mathbb{P}^1$ unchanged.



1st time : $\quad bl_x^* D = D' + E_x \Rightarrow (bl_x^* D)^2 = D'^2 + 2D'E_x + E_x^2$

$$\underset{D^2}{\overset{\parallel}{(bl_x^* D)^2}} = \underset{D'^2+1}{\overset{\parallel}{D'^2 + 2D'E_x + E_x^2}} \;\Downarrow\; D'^2 = n-1$$

$$D_0'^2 = D_0^2 = -n$$

2nd time :

$$\mathrm{cont}_{F'}^* D_0'' = D_0' + F'$$

$$(\mathrm{cont}_{F'}^* D_0'')^2 = (D_0' + F')^2$$

$$D_0''^2 = D_0'^2 + 2D_0'F' + F'^2 = D_0'^2 + 1 \Rightarrow D_0''^2 = -n+1$$

$$D''^2 = D'^2 = n-1$$

Since $\mathrm{elem}_x(\mathbb{F}_n)$ is one of the surfaces $\mathbb{F}_m$

$$\Rightarrow m = n-1.$$

$$\Rightarrow \boxed{\mathrm{elem}_x(\mathbb{F}_n) = \mathbb{F}_{n-1}.}$$

So we have

Prop | If $m \neq n$ are non-negative integers, then $\exists$ birational map $g : \mathbb{F}_n \dashrightarrow \mathbb{F}_m$ which is a composite of elementary transformations.

Castelnuovo's Rationality Criterion & its applications

THEOREM (Castelnuovo's rationality criterion)

S surface

then S is rational $\Leftrightarrow q(S) = P_2(S) = 0$.

Rmk $P_2(S) = 0 \Rightarrow p_g(S) = 0$

$\exists$ non-rational surfaces with $p_g = q = 0$, say

Enriques surfaces, Godeaux surfaces
(surf. of general type)

Def X n-dim'l variety

We say X is <u>unirational</u> if $\exists$ a dominant rational map

$$\mathbb{P}^n \dashrightarrow X$$

X is <u>rational</u> if $\exists$ a birational map

$$\mathbb{P}^n \overset{\sim}{\dashrightarrow} X$$

In other words,

X is rational if the rational functions field $K(X)$ is a pure transcendental extension of $\mathbb{C}$

X is unirational if $K(X) \subseteq$ a pure transcendental extension of $\mathbb{C}$.

Recall for curves

THEOREM (Lüroth)

$\forall$ unirational curve is rational.

Pf If C is unirational, $\Rightarrow \exists$ a surjective morphism $f: \mathbb{P}^1 \to C$

$\Rightarrow$ there is no non-zero holomorphic 1-form ω on C

$$\Big(f^* H^0(\Omega_C^1) \hookrightarrow \underset{\substack{\| \\ 0}}{H^0(\mathbb{P}^1, \Omega^1_{\mathbb{P}^1})} \Big)$$

Otherwise $f^*\omega$ is a nonzero holomorphic 1-form on $\mathbb{P}^1$ ↯

$\Rightarrow q(C) = 0$ and then C is rational.

Application of Castelnuovo's rationality criterion

Cor $\forall$ unirational surface is rational.

Pf let S be a unirational surface, then $\exists$ dominant rational map $\varphi: \mathbb{P}^2 \dashrightarrow S$.

By elemination of indeterminacy, $\exists$ a surj morph. $X \to S$

$$\begin{tikzcd} X \arrow[r] \arrow[d] & S \\ \mathbb{P}^2 \arrow[ur, dashed, "\varphi"'] & \end{tikzcd}$$

$\Rightarrow X$ rational surface $\xRightarrow{\text{Castelnuovo}} P_2(\cancel{X}) = q(X) = 0$.

$$\Rightarrow \quad q(S) = P_2(S) = 0 \xRightarrow{\text{Castelnuovo}} S \text{ rational}$$

$\square$

key Prop

> S minimal surface with $q = P_2 = 0$
> then $\exists$ smooth rational curve $C \subset S$ s.t. $C^2 \geq 0$

Granting this Proposition for the moment,

Proof of Castelnuovo's theorem. let $C \subset S$ be a smooth rat'l curve with $C^2 \geq 0$

$$0 \to \mathcal{O}_S(-C) \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{O}_C \longrightarrow 0 \quad \text{exact}$$

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \longrightarrow \mathcal{O}_C(C) \longrightarrow 0 \quad \text{exact}.$$

taking cohomology,

$$\longrightarrow H^0(\mathcal{O}_S(C)) \longrightarrow H^0(\mathcal{O}_C(C)) \longrightarrow \underset{\substack{\| \\ 0}}{H^1(\mathcal{O}_S)} \longrightarrow H^1(\mathcal{O}_S(C)) \longrightarrow \underset{\substack{\| \\ 0}}{H^1(\mathcal{O}_C(C))} \longrightarrow \underset{0 =}{H^2(\mathcal{O}_S)} \quad \text{exact}$$

C smooth rational

$$\underset{\substack{\| \\ 0}}{g(C)} = 1 + \frac{1}{2}(C^2 + KC)$$

$$\underset{\substack{\| \\ h^0(C)}}{\chi(\mathcal{O}_S(C))} = \chi(\mathcal{O}_S) + \frac{1}{2}(C^2 - KC) = 2 + C^2 \geq 2$$

let $D \in |C|$ be a divisor other than C.

the pencil generated by C and D has no fixed components.

$$|t_1 C + t_2 D| \qquad
\begin{tikzcd}
S \arrow[d, dashed, "f"'] & \hat{S} \arrow[l, "\varepsilon"'] \arrow[dl, "\hat{f}"] \\
{[t_1 : t_2] \in \mathbb{P}^1} &
\end{tikzcd}$$

f ← rational map, $\hat{f}$ ← morphism

blowing up base points of this linear pencil, we obtain a morphism
$\hat{f} : \hat{S} \longrightarrow \mathbb{P}^1$ with one fiber $\cong C$

by Noether-Enriques theorem $\Rightarrow$ S rational. $\square$

To prove the key Prop, we need

lemma 1

> S minimal surface with $K_S^2 < 0$.
> For all $a > 0$, $\Rightarrow \exists$ effective divisor D on S s.t.
> $$\begin{cases} K_S D \leq -a \\ |K_S + D| = \emptyset \end{cases}$$

Pf suffices to find an effective divisor $E \subset S$ s.t. $K_S E < 0$.

[Indeed, $\exists$ a component C of E s.t. $K_S C < 0$.

By genus formula, $\begin{cases} C^2 \geq -1 \\ C^2 = -1 \text{ only if } C \text{ is a } (-1)\text{-curve} \end{cases}$ ↯

$\Rightarrow C^2 \geq 0$.

$(aC + nK_S)C < 0$ for $n \gg 0$. $\Rightarrow |aC + nK| = \phi$ for $n \gg 0$.

$\Downarrow$

$\exists\, n \in \mathbb{Z}$ s.t. $\begin{cases} |aC + nK| \neq \phi \\ |aC + (n+1)K| = \phi \end{cases}$

If $D \in |aC + nK|$, then

$KD = aKC + nK^2 \underset{\Uparrow}{\leq} -a$

$KC < 0$ & $K^2 < 0$

&

$K + D \sim aC + (n+1)K \Rightarrow |K + D| = \phi$.]

Let H be a hyperplane section of S.

If $KH < 0$, can take $E = H$;

If $KH = 0$, R.-R. $h^0(K + nH) \geq \chi(K + nH) = \chi(\mathcal{O}_S) + \frac{1}{2}(K + nH)nH$

$\qquad = \frac{n^2}{2}H^2 + \chi(\mathcal{O}_S) > 0$ for $n \gg 0$.

$\Rightarrow |K + nH| \neq \phi$ for $n \gg 0$.

Can take $E \in |K + nH|$.

If $KH > 0$, set $r_0 = \frac{KH}{-K^2}$, then

$$\left.\begin{aligned} (H + r_0K)^2 &= H^2 + 2r_0HK + r_0^2K^2 \\ &= H^2 + \frac{(KH)^2}{-K^2} > 0 \\ (H + r_0K)K &= 0 \end{aligned}\right\} \Rightarrow$$

if $r \in \mathbb{Q}$

~~$r > r_0$~~ $r > r_0$

$r \to r_0$

(suff. close to r_0)

$\Downarrow$

$$\boxed{\begin{aligned} &(H + rK)^2 > 0 \\ &(H + rK)K < 0 \\ &(H + rK)H > 0 \end{aligned}}$$

If $r = \frac{p}{q} \in \mathbb{Q}$ where $p, q \in \mathbb{Z}_{>0}$. Put $D_m = mq(H + rK)$

then $D_m^2 > 0$ & $D_mK < 0$.

By Riemann-Roch,

$h^0(D_m) + h^0(K - D_m) = \chi(D_m) = \chi(\mathcal{O}_S) + \frac{1}{2}(D_m^2 - D_mK) \to +\infty \ (m \to \infty)$

Since $(K - D_m)H = KH - mq(H + rK)H < 0$ for $m \gg 0$. $\Rightarrow h^0(K - D_m) = 0$ for $m \gg 0$

take $E \in |D_m|$. $\Leftarrow |D_m| \neq \phi$ for $m \gg 0 \Leftarrow h^0(D_m) \to +\infty \ (m \to +\infty)$

□

back to <u>key Proposition</u>

> S minimal surface with $q = P_2 = 0$.
>
> then $\exists$ a smooth rational curve C on S s.t. $C^2 \geq 0$.

<u>pf</u> Step 1

> Suffices to show that $\exists$ eff. divisor D on S
>
> s.t. $KD < 0$, $|K+D| = \phi$.

For then some component C of D satisfies $\begin{cases} KC < 0 \\ |K+C| = \phi \end{cases}$

by R.-R..

$$0 = h^0(K+C) \geq \chi(K+C) = \underbrace{\chi(\mathcal{O}_S) + \frac{1}{2}(K+C)C}_{= g_a(C)}$$

$\Rightarrow$ C smooth rational curve

by genus formula,

$$\left.\begin{array}{l} \underbrace{g(C)}_{=0} = 1 + \frac{1}{2}\underbrace{(C^2 + KC)}_{=-2} \\ KC < 0 \end{array}\right\} \Rightarrow C^2 \geq -1$$

S minimal $\Rightarrow C^2 \neq -1 \Rightarrow C^2 \geq 0$.

Step 2 $K^2 < 0$ Case

<u>Key Prop</u> follows from Step 1 & above lemma.

Step 3 $K^2 = 0$ Case

$$h^0(-K) - h^1(-K) = \chi(-K) \overset{R.R.}{=} \chi(\mathcal{O}_S) + \frac{1}{2}(-K)(-2K) = 1 + K^2$$

$$h^0(-K) - h^0(2K) = h^0(-K) \qquad 1$$

$\Rightarrow |-K| \neq \phi$

let H be a hyperplane section of S. then H ample

$$\cancel{h^0(H+nK) \geq \chi(H+nK) = \chi(\mathcal{O}_S) + \frac{1}{2}(H+nK)(H+(n-1)K)} = \cancel{1 + \frac{2n-1}{2}HK + \frac{1}{2}H^2}$$

$|-K| \neq \phi \Rightarrow HK < 0$ $\quad |H-K|$

$\Downarrow$

$(H+nK)H = H^2 + nKH < 0$ for $n \gg 0$

$\Downarrow$

$\exists n \geq 0$ s.t. $|H+nK| \neq \phi$ & $|H+(n+1)K| = \phi$.

let $D \in |H+nK|$, then $|K+D| = \phi$.

$K.D = K(H+nK) = KH < 0$

by Step 1, <u>Key Prop</u> holds.

Step 4 $K^2 > 0$ case.

$$h^0(-K) \geq \chi(-K) = \chi(\mathcal{O}_S) + \frac{1}{2}(-K)(-2K)$$
$$= 1 + K^2 \geq 2$$

- Suppose that $|-K|$ contains a reducible divisor, say $D \in |-K|$, $D = A+B$ $(A \geq 0, B \geq 0)$

$$DK = -K^2 < 0 \Rightarrow KA < 0 \text{ or } KB < 0$$
$$DK = KA + KB$$

for example $KA < 0$. then $|K+A| = |-B| = \phi$ (negative div)

by Step 1 again, Key Prop holds.

- From now on, WMA every divisor in $|-K|$ is irreducible.

$$(H+nK)H = H^2 + nKH < 0 \text{ for } n \gg 0.$$
$$\Downarrow$$
$$\exists\, n > 0 \text{ s.t. } |H+nK| \neq \phi \ \& \ |H+(n+1)K| = \phi.$$

Step 5 Suppose we can find H, n as above s.t. $H+nK \not\equiv 0$.

let $E \in |H+nK|$, say $E = \sum n_i C_i \geq 0$.

then

$$KE = -D.E$$

$D \in |-K|$ irreducible & $D^2 = K^2 > 0 \Rightarrow DE \geq 0$, $DE = -KE$

i.e. $KE \leq 0$

$\Downarrow$

$$0 = h^0(K+C_i) \geq \chi(K+C_i) \Leftarrow |K+C_i| = \phi \Leftarrow KC_i < 0 \text{ for some } C_i$$
$$\chi(K+C_i) = \chi(\mathcal{O}_S) + \frac{1}{2}(K+C_i)C_i = g(C_i)$$

$\Downarrow$

$$g(C_i) = 0 \ \& \ C_i^2 = -2 - KC_i$$

If $KC_i \leq -2$, then $C_i^2 \geq 0$ i.e. Key Prop. holds

If $KC_i = -1$, then $C_i^2 = -1$ & $g(C_i) = 0$ $\lightning$ (S minimal)

If $KC_i = 0$, then $C_i^2 = -2$.

$$|-K| \neq \phi \Rightarrow 0 = h^0(K+C_i) \geq h^0(2K+C_i) \Rightarrow h^0(2K+C_i) = 0$$

Calculate $h^0(-K-C_i)$ via Riemann-Roch,

$$h^0(-K-C_i) = \chi(-K-C_i) = \chi(\mathcal{O}_S) + \tfrac{1}{2}(-K-C_i)(-2K-C_i)$$
$$= 1 + \tfrac{1}{2}\left[(K+C_i)^2 + K(K+C_i)\right]$$
$$= 1 + \tfrac{1}{2}(2K^2 + 3KC_i + C_i^2)$$
$KC_i = 0$, $C_i^2 = -2$ $\qquad = K^2$

$\Rightarrow\ h^0(-K-C_i) = K^2 \geq 1$.

$$\left.\begin{array}{l} C_i^2 = -2 \\ K^2 > 0 \end{array}\right\} \Rightarrow C_i \not\equiv K \Rightarrow \exists \text{ non-zero effective divisor } A$$
such that $A + C_i \in |-K|$

$\Downarrow$

$|-K|$ contains a reducible divisor $\lightning$

Step 6 It remains to consider the case where $\operatorname{Pic} S \cong \mathbb{Z}[K]$

that is, every effective divisor is a multiple of K.

From the exponential exact seq.

$$0 \to \mathbb{Z} \to \mathcal{O}_S \to \mathcal{O}_S^* \to 0$$

taking cohomology, we have exact seq.

$$\underset{\underset{0}{\parallel}}{H^1(S,\mathcal{O}_S)} \longrightarrow \operatorname{Pic} S \xrightarrow{c_1} H^2(S,\mathbb{Z}) \longrightarrow \underset{\underset{0}{\parallel}}{H^2(\mathcal{O}_S)}$$

$$\Rightarrow \underset{\underset{\mathbb{Z}[K]}{\cong}}{\operatorname{Pic} S} \cong H^2(S,\mathbb{Z}) \Rightarrow b_2(S) = 1$$

By Poincaré duality, the intersection form on $H^2(S,\mathbb{Z})$ is unimodular

$\Downarrow$

$K^2 = 1$

By Noether formula

$$1 = \chi(\mathcal{O}_S) = \tfrac{1}{12}(K^2 + e(S))$$
$$= \tfrac{1}{12}(K^2 + 2 - 2b_1 + b_2) \Rightarrow b_1 = -4 \ \lightning$$

$\square$

Minimal models of rational surfaces

THEOREM

Let S minimal rational surface,

then $S \cong \mathbb{P}^2$ or $\mathbb{F}_n$ ~~with~~ (for $n \geq 0$ & $n \neq 1$).

Pf. let H be a hyperplane section of S. Consider the set

$\Sigma := \{$ Smooth rational curves $C \subset S$ with $C^2 \geq 0\}$

By Key Prop. $\Sigma \neq \phi$

let $m = \min\{C^2 \mid C \in \Sigma\}$, then

if let $\Sigma_m := \{C \in \Sigma \mid C^2 = m\}$, we can choose a curve $C \in \Sigma_m$

with $H.C$ minimal for $\forall C \in \Sigma_m$.

Step 1 Claim: $\forall$ divisor $D \in |C|$ is a smooth rational curve.

put $D = \sum n_i C_i \geq 0$.

- by genus formula, $\underset{\overset{\|}{0}}{g(C)} = 1 + \frac{1}{2}(KC + C^2) \Rightarrow KC + C^2 = -2$.

$0 = h^0(K+C_i) \geq \chi(K+C_i) \underset{\overset{\|}{p_a(C_i)}}{} \Leftarrow h^0(K+C_i) = 0$ for $\forall i$ $\Leftarrow \overset{h^0(K+D)}{\overset{\|}{h^0(K+C) = 0}} \xLeftarrow{C^2 \geq 0}$

(each C_i is a sm rat'l curve.)

- C smooth rational curve $\xRightarrow{\text{genus formula}} (K+C)C = -2 \xRightarrow{C^2 \geq 0} \overset{KD}{\overset{\|}{KC}} < 0$

$\Downarrow$

$KC_i < 0$ for some i

$C_i^2 \geq 0 \Longleftarrow KC_i + C_i^2 = -2$

(if $KC_i = -1$, then $C_i^2 = -1$ & S minimal $\Rightarrow KC_i \leq -2$)

- Put $D' := \sum_{i \neq j} n_j C_j$, then $D = n_i C_i + D'$, $D'C_i \geq 0$

$C^2 = D^2 = (n_i C_i + D')^2 = n_i^2 C_i^2 + 2n_i C_i D' + D'^2$

$= n_i^2 C_i^2 + n_i C_i D' + D.D'$

~~$C_i^2 \geq 0$~~ $\left.\begin{matrix} D' \geq 0 \\ C^2 \geq 0 \end{matrix}\right\} \Rightarrow \underset{DD'}{\overset{\|}{D'C}} \geq 0 \Longrightarrow m = C^2 \geq n_i^2 C_i^2 \geq 0$

C_i smooth rational

By minimality of m,

$C_i^2 = m$ & $n_i = 1$.

By the choice of C ($C \in \Sigma_m$ with HC minimal),

$HC = HD = HC_i + HD' \Rightarrow HD' = 0 \xRightarrow[H \text{ ample}]{} D' = 0 \Rightarrow D = C_i$

that is, every divisor $D \in |C|$ is a sm rat'l curve.

Step 2 Show that $\dim|C| \leqslant 2$.

let $p \in S$. Consider the exact seq.

$$0 \to \mathfrak{m}_p/\mathfrak{m}_p^2 \longrightarrow \mathcal{O}_{p,S}/\mathfrak{m}_p^2 \longrightarrow \mathcal{O}_p/\mathfrak{m}_p \to 0$$

$\mathfrak{m}_p/\mathfrak{m}_p^2$: 2-dim'l; $\mathcal{O}_p/\mathfrak{m}_p \cong k(p) \cong \mathbb{C}$, 1-dim'l

then $\dim_{\mathbb{C}} \mathcal{O}_p/\mathfrak{m}_p^2 = 3$

$\Downarrow$

the linear system of curves of $|C|$ passing through p with multiplicity $\geqslant 2$

$$P := \{ \text{curves } D \in |C| \ : \ \mathrm{mult}_p(D) \geqslant 2 \}$$

has $\dim \geqslant \left[(h^0(C)-1) - \frac{2(2+1)}{2} \right]$ in $|C| = \mathbb{P}^{h^0(C)-1}$

i.e. Codim $\leqslant 3$ in $|C|$.

If $\dim|C| \geqslant 3$, then $P \neq \phi$. $\lightning$ (by Step 1, every divisor D in $|C|$ is a smooth rational curve $\Rightarrow \mathrm{mult}_p(D) = 1$)

Step 3 Consider the map defined by $|C|$.

- let $C_0 \in |C|$. Consider the short exact sequence

$$0 \to \mathcal{O}_S \longrightarrow \mathcal{O}_S(C) \longrightarrow \mathcal{O}_{C_0}(C_0) \longrightarrow 0 \qquad \mathcal{O}_{C_0}(C_0) \cong \mathcal{O}_{C_0}(m)$$

$C_0^2 = C^2 = m$

taking cohomology

$$0 \to H^0(\mathcal{O}_S) \longrightarrow H^0(C) \longrightarrow H^0(\mathcal{O}_{C_0}(m)) \longrightarrow H^1(\mathcal{O}_S) \quad \text{exact}$$

$H^0(\mathcal{O}_S) \cong \mathbb{C}$, $\dim = 1$; $\dim H^0(\mathcal{O}_{C_0}(m)) = \binom{m+1}{1} = m+1$; $H^1(\mathcal{O}_S) = 0$

$\Rightarrow h^0(C) = m+2$

$$\deg \mathcal{O}_S(C)|_{C_0} = C^2 = m \geqslant 2 \times \overset{g(C_0)}{0} = 0$$

$\Rightarrow |C||_{C_0}$ has no base points $\Rightarrow |C|$ has no base points.

by Step 2. $\underset{m+1}{\dim|C|} \leqslant 2 \Rightarrow m = 0$ or 1.

<u>Case $m=0$.</u> $|C|$ defines a morphism $f: S \longrightarrow \mathbb{P}^1$ & all fibers of f are smooth rational curves in $|C|$.

$\Rightarrow S$ geometrically ruled surface over $\mathbb{P}^1$.

$\Rightarrow S \cong \mathbb{F}_n$

S minimal $\Rightarrow n \neq 1$

Case $m=1$. $|C|$ defines a morphism $f: S \longrightarrow \mathbb{P}^2$.

for each $x \in \mathbb{P}^2$.

$f^{-1}(x) =$ intersection of two distinct rational curves in $|C|$ ·

$C^2 = m = 1$

a single point

$\Rightarrow$ $f: S \longrightarrow \mathbb{P}^2$ is an isom.

i.e. $S \cong \mathbb{P}^2$.

$\square$