Ruled surfaces

Def S : surface

S is <u>ruled</u> if S is birationally equivalent to $C \times \mathbb{P}^1$ where C is a smooth curve.

S is <u>rational</u> if $C = \mathbb{P}^1$.

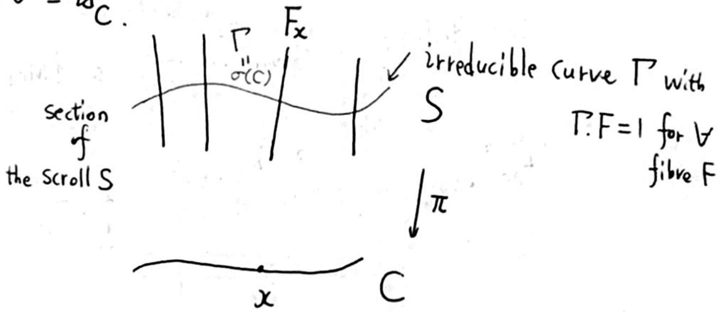
S is <u>geometrically ruled surface</u> over C if $\exists$ surjective morphism $\pi : S \longrightarrow C$ onto a smooth curve C (called base curve) such that all fibres $F_x = \pi^{-1}(x)$ isomorphic to $\mathbb{P}^1$ for $\forall x \in C$.

also called $\mathbb{P}^1$-bundle or a scroll over C; these fibres called rullings of S

S is <u>rationally ruled</u> if it is both rational & ruled.

Remark A priori, for a geometrically ruled surface $\pi : S \longrightarrow C$, π admits a section, that is, a morphism $\sigma : C \longrightarrow S$ such that $\pi \circ \sigma = id_C$.

$$\begin{tikzcd} S \arrow[d, "\pi"'] \\ C \arrow[u, bend right, "\sigma"'] \end{tikzcd}$$



geometrically ruled (surfaces) $\Longrightarrow$ ruled.

THEOREM (Noether–Enriques)

S : surface

$p : S \longrightarrow C$ morphism onto a smooth curve C

Suppose $\exists\, x \in C$ s.t. $\begin{cases} p \text{ smooth over } x \\ \& \\ F_x = p^{-1}(x) \cong \mathbb{P}^1 \end{cases}$

then $\exists$ a Zariski open subset $U \subseteq C$ containing x & an U-isomorphism

$$\begin{tikzcd} p^{-1}(U) \arrow[rr, "\cong"] \arrow[dr, "p"'] & \arrow[d, phantom, "\circlearrowright"] & U \times \mathbb{P}^1 \arrow[dl, "pr_1"] \\ & U & \end{tikzcd}$$

In particular, S is ruled.

[i.e. p locally trivial fibration with fibre $\mathbb{P}^1$ (that is, $\mathbb{P}^1$-bundle) (in the Zariski topology)]

Pf Step 1 : $H^2(S, \mathcal{O}_S) = 0$

Put $F = p^{-1}(x)$

then $F^2 = 0$ (since F is a fibre)

by genus formula $\Rightarrow K_S . F = -2$

Suppose that $H^2(S, \mathcal{O}_S) \neq 0$, then $h^0(K_S) = h^2(\mathcal{O}_S) \geq 1$

& hence $|K_S|$ contains an effective divisor D.

One has

$$\left.\begin{array}{l} K_S F = -2 \\ \quad \shortparallel \\ \quad DF \\ F^2=0,\ D\geqslant 0 \Rightarrow DF \geqslant 0 \end{array}\right\} \lightning$$

Step 2 : | $\exists$ a section $\sigma: C \longrightarrow S$ of p, that is, a curve $\overset{\sigma(C)}{\overset{\shortparallel}{H}}$ on S with $H \twoheadrightarrow C$ & $HF=1$ for each fibre F.

$$\left.\begin{array}{l} p_* \mathcal{O}_S \text{ torsion-free coherent sheaf on } C\ (\mathcal{O}_C\text{-module}) \\ C \text{ smooth curve} \end{array}\right\} \Rightarrow p_*\mathcal{O}_S \text{ locally free of finite rank}$$

$$\left.\begin{array}{l} p \text{ flat} \\ h^1(p^{-1}(x), \mathcal{O}_{p^{-1}(x)})=0 \end{array}\right\} \xRightarrow{\text{semi-continuity}} h^1(p^{-1}(x'), \mathcal{O}_{p^{-1}(x')})=0$$

in a neighborhood V of x in C where $\forall x' \in V$

$\Downarrow$ Grauert

$R^1 p_* \mathcal{O}_S$ locally free on V & $R^1 p_* \mathcal{O}_S \otimes k(x') \xrightarrow{\simeq} H^1(\mathcal{O}_{p^{-1}(x')})$ for $\forall x' \in V$

$\Downarrow$ Cohomology & base-change theorem

$p_* \mathcal{O}_S \otimes k(x') \xrightarrow{\simeq} H^0(\mathcal{O}_{p^{-1}(x')})$ for $\forall x' \in V$.

at the point $x \in C$, $p^{-1}(x) \cong \mathbb{P}^1$

$\Rightarrow h^0(p^{-1}(x), \mathcal{O}_{p^{-1}(x)}) = 1$

$\Rightarrow$ the locally free sheaf $p_*\mathcal{O}_S$ is of rank 1

$\Rightarrow p_*\mathcal{O}_S = \mathcal{O}_C$ & $K(C)$ is algebraically closed in $K(S)$

| **Lemma**
If $f: X \longrightarrow Y$ dominant morphism with X smooth irred. var. of $\dim \geqslant 2$, Y irred. curve
& $K(Y)$ is algebraically closed in $K(X)$
then the fibre $f^{-1}(y)$ is geometrically integral for all y but a finite number of closed points $y \in Y$.

$\Rightarrow$ $\exists$ an open neighborhood $U_1 \subseteq V$ of x in C such that the fibre $F_{x'} = p^{-1}(x')$ is geometrically integral for $\forall x' \in U_1$.

$\Downarrow$ p flat

the Hilbert polynomial $P_{x'} \in \mathbb{Q}[z]$ of $F_{x'}$ is independent of $x' \in U_1$

$\Downarrow$

the arithmetic genus of $F_{x'}$, $p_a(F_{x'}) := (-1)^{\dim F_{x'}}(P_{x'}(0)-1)$ is independent of $x' \in U_1$

Note $p_a(F_x) = p_a(\mathbb{P}^1) = 0$

$\Rightarrow$ the generic fibre F_η has arithemetic genus 0

& $F_{x'} \cong \mathbb{P}^1$ for $\forall\, x' \in U_1 \subseteq V \subseteq C$

$$\begin{tikzcd} F_\eta \arrow[r] \arrow[d] & S \arrow[d, "p"] \\ \operatorname{Spec} k(\eta) \arrow[r] & C \end{tikzcd}$$

$\Rightarrow F_\eta \cong \mathbb{P}^1_{k(\eta)} = \mathbb{P}^1_{k(C)} \xhookrightarrow{|-K_{F_\eta}|} \mathbb{P}^2_{k(C)}$

as a conic

$(Q\ (x_0, x_1, x_2) = \sum l_{ij}(x)\, x_i x_j) \subset \mathbb{P}^2 \times \mathbb{A}^1$
$\qquad\qquad\qquad\qquad\quad \| $
$\qquad\qquad\qquad\qquad\quad 0$

with $l_{ij} \in k[x]$

THEOREM (Tsen)

K function field of a curve C over $\mathbb{C}$

$X \subset \mathbb{P}^n_K$ any hypersurface of deg $d \leq n$ over K

then X has a K-rational point.

i.e. K is quasi-algebraically closed (C_1)

by Tsen's theorem,

F_η has a $k(C)$-rational point

$$\begin{array}{c} k(C) \\ \| \\ \mathcal{O}_{\eta, C} \end{array}$$

that is, $\exists$ a $k(C)$-morphism $\sigma' : \operatorname{Spec} k(C) \longrightarrow F_\eta$

$$\begin{array}{c} \operatorname{Spec} k(C) \\ \| \\ \operatorname{Spec}(\mathcal{O}_{\eta, C}) \end{array}$$

$$\begin{tikzcd} \operatorname{Spec} k(C) \arrow[r, "\sigma'"] \arrow[d, equal] & F_\eta \arrow[r, "p"] & C \\ \operatorname{Spec} \mathcal{O}_{\eta, C} \arrow[urr, bend right, "\text{Canonical inclusion}"'] & & \end{tikzcd}$$

& $\operatorname{Spec} k(C) \xrightarrow{\sigma'} F_\eta \xrightarrow{p} C$, $\circlearrowleft$

$\Rightarrow \exists$ an open nbhd W of η such that ($W \subset C$)

$$\begin{tikzcd} W \arrow[r] \arrow[rr, bend right, "\mathrm{id}_W"'] & p^{-1}(W) \arrow[r, "p"] & W \end{tikzcd}$$

i.e. $\exists$ a rational section $C \overset{\sigma}{\dashrightarrow} S$ of p.

$\left.\begin{array}{l} C \text{ smooth curve} \\ S \text{ projective variety} \end{array}\right\} \Rightarrow$ the rational map $C \overset{\sigma}{\dashrightarrow} S$ is an everywhere defined morphism. $\sigma : C \longrightarrow S$.

Now put $H = \sigma(C) \subset S$, then $H \twoheadrightarrow C$ &

$H . F_{x'} = 1$ for $\forall\, x' \in C$.

Step 3 (local structure of $p : S \to C$)

$$\begin{tikzcd} p^{-1}(U_1) \arrow[r, "\cong"] \arrow[dr, "p"'] & \mathbb{P}((p|_{U_1})_* \mathcal{O}(H)) \arrow[d, "\pi"] \\ & U_1 \end{tikzcd}$$

For convenience, let $S_1 := p^{-1}(U_1)$ & $p_1 := p|_{p^{-1}(U_1)} : S_1 \longrightarrow U_1$

$\left.\begin{array}{l} \text{all fibres of } p_1 \text{ are projective lines} \\ \mathcal{O}_{S_1}(H) \otimes \mathcal{O}_{F_{x'}} \cong \mathcal{O}_{F_{x'}}(1) \text{ for } \forall\, x' \in U_1 \end{array}\right\} \Rightarrow h^0(\mathcal{O}_S(H) \otimes k(x')) = 2$ (as $k(x')$-vector space) for $\forall\, x' \in U_1$

Again by cohomology & base change theorem,

$$E := p_{1*}\mathcal{O}_{S_1}(H) \text{ is a locally free rk 2 } \mathcal{O}_{U_1}\text{-module}.$$

&

$$p_{1*}\mathcal{O}_{S_1}(H) \otimes k(x') \xrightarrow{\cong} H^0(\mathcal{O}_{S_1}(H) \otimes \mathcal{O}_{F_{x'}})$$

for $\forall\, x' \in U_1$.

$$\Rightarrow \underset{\substack{\| \\ p_1^* E}}{p_1^* p_{1*} \mathcal{O}_{S_1}(H)} \twoheadrightarrow \mathcal{O}_{S_1}(H)$$

By the universal property of $\mathbb{P}(E)$,

$$\begin{tikzcd} & \mathbb{P}(E) \arrow[d] \\ S_1 \arrow[ur, "\exists\varphi"] \arrow[r, "p_1"'] & U_1 \end{tikzcd}$$

s.t. $\mathcal{O}_{S_1}(H) \cong \varphi^* \mathcal{O}_{\mathbb{P}(E)}(1)$.

Over any point $x' \in U_1$,

$$\begin{array}{ccc} F_{x'} & \xrightarrow{\cong} & \mathbb{P}^1_{k(x')} \\ \cap & & \cap \\ S_1 & \xrightarrow{\varphi} & \mathbb{P}(E) \end{array}$$

$\Rightarrow \varphi$ is an isom. $\square$

If we strengthen the assumptions on the Noether-Enriques theorem by requiring that all fibres are isomorphic to $\mathbb{P}^1$, that is, S is a geometrically ruled surface.

then we can build the following connection between geom. ruled surfaces over a base curve C & rank 2 vector bundles$/C$

More precisely,

<u>Prop</u>. Let S be a geometrically ruled surface over a smooth curve C then $\exists$ a locally free rank 2 $\mathcal{O}_C$-module E & an isomorphism $\varphi: S \longrightarrow \mathbb{P}_C(E)$ over C.

Furthermore, for two rank 2 vector bundles E, E' on C, the bundles $\mathbb{P}_C(E) \longrightarrow \mathbb{P}_C(E')$ are C-isomorphic $\Leftrightarrow \exists$ a line bundle L on C s.t.

$$E' \cong E \otimes_{\mathcal{O}_C} L.$$

<u>Pf</u> By the proof of theorem of Noether-Enriques, suffices to show the 2nd statement.

If $E' \cong E \otimes L$, then $\mathrm{Sym}^n E' \cong \mathrm{Sym}^n(E \otimes L) \cong \mathrm{Sym}^n(E) \otimes L^{\otimes n}$

$$(x_1 \otimes y_1)\cdots(x_n \otimes y_n) \longmapsto (x_1 x_2 \cdots x_n) \otimes (y_1 \otimes \cdots \otimes y_n)$$

$\Rightarrow S_{\mathcal{O}_C}(E') \cong \bigoplus_{n \geq 0} \mathrm{Sym}^n(E) \otimes L^{\otimes n}$ as graded $\mathcal{O}_C$-alg.

So suffices to show that $\mathrm{Proj}\, S_{\mathcal{O}_C}(E) \xrightarrow{\sim} \mathrm{Proj} \bigoplus_{n\geq 0} (\mathrm{Sym}^n E)\otimes L^{\otimes n}$ over C.

the question is local on C

$C = \bigcup_i U_i$ $\quad$ $\underbrace{U_i \text{ open}}_{\text{affine}}$ subset of C $\quad$ $L|_{U_i} \cong \mathcal{O}_{U_i}$. over $U_i = \mathrm{Spec}\, A$ with a singler generator c

then $\forall n>0$, $\mathrm{Sym}^n E \longrightarrow \mathrm{Sym}^n E \otimes L^{\otimes n}$ is an A-module isom.

$x_n \longmapsto x_n \otimes c^{\otimes n}$

$\Rightarrow$ $\mathbb{N}$-graded alg. isom. $\varphi_c : S(E) \longrightarrow \bigoplus_{n\geq 0} \mathrm{Sym}^n E \otimes L^{\otimes n}$

Now let $f \in S(E)_+$ be a form of degree d, then

for $\forall$ $x \in S(E)_{nd}$, $\forall$ invertible element $\varepsilon \in A$,

$$\frac{(x\otimes c^{nd})}{(f\otimes c^d)^n} = \frac{(x \otimes (\varepsilon c)^{nd})}{(f\otimes(\varepsilon c)^d)^n}$$

$\Rightarrow$ $\varphi_c : S(E)_{(f)} \xrightarrow{\sim} \left(\bigoplus_{n\geq 0} \mathrm{Sym}^n E \otimes L^{\otimes n}\right)_{(f\otimes c^d)}$

independent of the choice of generator of L.

$\Longrightarrow$ $\mathbb{P}(\underset{E\otimes L}{\overset{\rotatebox{90}{$\cong$}}{E'}}) \overset{\theta}{\underset{\sim}{\longleftrightarrow}} \mathbb{P}(E)$ over C & $\theta^* \mathcal{O}_{\mathbb{P}(E\otimes L)}(n) \cong \mathcal{O}_{\mathbb{P}(E)}(n)\otimes L^{\otimes n}$

(above argument holds/works on arbitrary schemes)

Conversely, if we have a C-isomorphism

$$\begin{tikzcd} \mathbb{P}(E) \arrow[rr, "\theta", "\sim"'] \arrow[dr, "\pi"'] & & \mathbb{P}(E') \arrow[dl, "\pi'"] \\ & C & \end{tikzcd}$$

a priori, we know that

<u>Prop</u> | X arbitrary alg var.
E locally free $\mathcal{O}_X$-module of finite rank,
$f : \mathbb{P}(E) \to X$ the projective fibre bundle on X
then $\mathrm{Pic}(\mathbb{P}(E)) \cong \mathbb{Z}[\mathcal{O}_{\mathbb{P}(E)}(1)] \oplus f^*(\mathrm{Pic}X)$

$\Rightarrow$ $\theta^* \mathcal{O}_{\mathbb{P}(E')}(1) \cong \pi^* L \otimes \mathcal{O}_{\mathbb{P}(E)}(m)$

for some line bundle $L \in \mathrm{Pic}(C)$ & $m \in \mathbb{Z}$.

By projection formula,

$$\pi_* \theta^* \mathcal{O}_{\mathbb{P}(E')}(1) \cong \pi_* (\pi^* L \otimes \mathcal{O}_{\mathbb{P}(E)}(m))$$

$\pi_* \theta^* \mathcal{O}_{\mathbb{P}(E')}(1) = \pi'_*(\mathcal{O}_{\mathbb{P}(E')}(1)) = E'$

$\pi_* (\pi^* L \otimes \mathcal{O}_{\mathbb{P}(E)}(m)) \overset{\text{P.F.}}{=} L \otimes \pi_* \mathcal{O}_{\mathbb{P}(E)}(m) = \begin{cases} L\otimes S^m E & (m\geq 0) \\ 0 & (m<0) \end{cases}$

$\Rightarrow$ $m = 1$

i.e. $E' \cong L \otimes E$.

Cor $p: S \to C$ geometrically ruled surface

$\sigma: C \to S$ a section of p

Put $H = \sigma(C)$, F a fibre of p

then $\operatorname{Pic} S \cong \mathbb{Z}\cdot h \oplus p^* \operatorname{Pic}(C)$

$\operatorname{Pic} S/\equiv_{num} = \operatorname{Num} S \cong \mathbb{Z}\cdot h \oplus \mathbb{Z}\cdot f$

$H^2(S, \mathbb{Z}) \cong \mathbb{Z} h \oplus \mathbb{Z} f$

where h, f denote class of H, F, resp in the corresponding group

Pf. $S = \mathbb{P}(E)$ for some rank 2 vector bundle E on C

then $\mathcal{O}_S(H) \cong \mathcal{O}_{\mathbb{P}(E)}(1) \otimes p^* L'$ for some $L' \in \operatorname{Pic} C$.

note that $\operatorname{Pic}(\mathbb{P}(E)) \cong \mathbb{Z}[\mathcal{O}_{\mathbb{P}(E)}(1)] \oplus p^* \operatorname{Pic}(C)$

$\qquad \| $

$\operatorname{Pic} S$

Consider the exponential exact seq.

$$0 \to \mathbb{Z} \to \mathcal{O}_S \to \mathcal{O}_S^* \to 0$$

taking cohomo.

by Step 1 of proof of Noether-Enriques

$$0 \to H^1(S, \mathbb{Z}) \to H^1(S, \mathcal{O}_S) \to \underset{\operatorname{Pic} S}{\overset{\|}{H^1(\mathcal{O}_S^*)}} \xrightarrow{c_1} H^2(S, \mathbb{Z}) \to \underset{0}{\overset{\|}{H^2(\mathcal{O}_S)}}$$

$\Rightarrow H^2(S, \mathbb{Z})$ is a quotient of $\operatorname{Pic} S$.

& two points of C have the same cohomology class in $H^2(C, \mathbb{Z}) \cong \mathbb{Z}$

$f^2 = 0$, $fh = 1 \Rightarrow$ two linearly independent cohomology classes f, h generate $H^2(S, \mathbb{Z})$.

$\operatorname{Num}(C) = \mathbb{Z}\{x\}$ for arbitrary closed point $x \in C$.

$\Rightarrow p^* \operatorname{Num}(C) \cong \mathbb{Z}\cdot f$

&

hence $\operatorname{Num}(S) \cong \mathbb{Z} h \oplus \mathbb{Z} f$.

$\square$

Vector bundles on Curves & Extensions of Vector bundles

Prop [Splitting of vector bundles]
C smooth projective curve.

E locally free $\mathcal{O}_C$-module of rank $r \geq 1$

then $\exists$ a chain of locally free subsheaves of E

$$0 = E_0 \subset E_1 \subset \cdots \subset E_r = E$$

with $\begin{cases} \operatorname{rank} E_i = i \\ E_i / E_{i-1} \text{ an invertible } \mathcal{O}_C\text{-module for } \forall\, 1 \leq i \leq r. \end{cases}$

pf Use induction on $\operatorname{rank} E = r$.

Suffices to show that E has an invertible subsheaf L such that E/L is locally free of rank $r-1$

let $P = \mathbb{P}(E^\vee)$ be the projective bundle over C with $\pi : P \to C$ the canonical projection.

then $\pi^* E^\vee \twoheadrightarrow \mathcal{O}_P(1)$

$\Rightarrow \mathcal{O}_P(-1) \hookrightarrow \pi^* E$

the quotient $E' = \pi^* E / \mathcal{O}_P(-1)$ is locally free

let $s \in \Gamma(U, E)$ be a nonzero section of E over certain nonempty open subset $U \subseteq C$.

s defines a morphism of sheaves $\quad \iota : U \hookrightarrow C$ inclusion

$$E^\vee|_U = \iota^* E^\vee \longrightarrow \mathcal{O}_U$$

it is surjective over some nonempty open $V \subseteq U$

$$i^* E^\vee \twoheadrightarrow \mathcal{O}_V \qquad i : V \hookrightarrow C \text{ inclusion}$$

$\Rightarrow$ we have a morphism $V \xrightarrow{i_s} P$

$$\begin{tikzcd} & P = \mathbb{P}(E^\vee) \arrow[d, "\pi"] \\ V \arrow[ur, "i_s"] \arrow[r, hook, "i"'] \arrow[ur, phantom, "\circlearrowleft", shift right=3] & C \end{tikzcd}$$

$\Rightarrow i_s : C \dashrightarrow P$ is a rational section of π

C smooth curve, P projective var $\Rightarrow i_s : C \longrightarrow P$ (everywhere defined) a regular section of π

$$0 \to \mathcal{O}_P(-1) \to \pi^* E \longrightarrow E' \longrightarrow 0 \quad \text{exact}$$

$$\Rightarrow 0 \to \underbrace{i_s^* \mathcal{O}_P(-1)}_{= L_s} \to i_s^* \pi^* E \longrightarrow i_s^* E' \to 0 \quad \text{exact seq. of}$$

locally ~~of~~ free $\mathcal{O}_C$-modules.

invertible subsheaf constructed from a nonzero section

$\square$

Remark the chain $0=E_0\subset E_1\subset E_2\subset\cdots\subset E_r=E$

with E_i locally free of rank i

$E_i/E_{i-1}=L_i$ line bundle for $\forall\, 1\le i\le r$

called the <u>splitting</u> of E, denoted by $(L_1,\dots,L_r)$

If E locally free of rank $r\geq 1$ on a curve C

write $\det E$ for $\Lambda^r E$

$\deg E$ for $\deg(\det E)$

For $\forall$ short exact sequence of locally free sheaves

$$0\to \underset{\mathrm{rk}\, n'}{\mathcal{F}'}\to \underset{\mathrm{rk}\, n}{\mathcal{F}}\to \underset{\mathrm{rk}\, n''}{\mathcal{F}''}\to 0$$

we have $\Lambda^n\mathcal{F}\cong\Lambda^{n'}\mathcal{F}'\otimes\Lambda^{n''}\mathcal{F}''$

$\Rightarrow$ deg is additive w.r.t. short exact seq. of vector bundles

So if $(L_1,\dots,L_r)$ is a splitting of E, then $E_1=L_1$

$$\left.\begin{array}{l} 0\to E_1\to E_2\to E_2/E_1 = L_2\to 0\\ 0\to E_2\to E_3\to L_3\to 0\\ \quad\vdots\\ 0\to E_{r-1}\to E_r\to L_r\to 0\end{array}\right\}\Rightarrow \begin{array}{l}\deg E_1=\deg L_1\\ \deg E_2=\deg E_1+\deg L_2\\ \deg E_3=\deg E_2+\deg L_3\\ \quad\vdots\\ \deg E=\deg E_{r-1}+\deg L_r\end{array}$$

$$\Rightarrow \deg E=\deg L_1+\deg L_2+\dots+\deg L_r=\sum_{i=1}^r\deg L_i.$$

Riemann-Roch theorem

If E locally free $\mathcal{O}_C$-module of rank r on a smooth projective curve C

then $\chi(E)=\deg E+r(1-g(C))$

pf Induction on r.

- $r=1$, this is the usual R.-R. for line bundles/divisors
- assume $r>1$. then $\exists$ s.e.s. of locally free $\mathcal{O}_C$-modules

$$0\to \underset{\mathrm{rk}=1}{L}\to E\to \underset{\mathrm{rk}=r-1}{E'}\to 0$$

$$\begin{aligned}\Rightarrow\quad \chi(E)&=\chi(L)+\chi(E')\\ &\overset{\substack{\text{usual R.-R.}\\ \text{inductive hypothesis}}}{=}[\deg L+(1-g)]+[\deg E'+(r-1)(1-g)]\\ &=\deg E+r(1-g)\end{aligned}$$

$\square$

Extensions of locally free sheaves / vector bundles

X complete alg variety

E_1, E_2 two locally free $\mathcal{O}_X$-modules of finite rank

Def an extension of E_2 by E_1 is an exact seq. of the form

$$0 \to E_1 \xrightarrow{\varphi} E \xrightarrow{\psi} E_2 \longrightarrow 0$$

with E locally free of finite rank
(automatically holds)

If $0 \to E_1 \xrightarrow{\varphi'} E' \xrightarrow{\psi'} E_2 \to 0$ is another extension of E_2 by E_1. we say that the two extensions are isomorphic if $\exists$ an $\mathcal{O}_X$-module homo. $\theta : E \to E'$ such that the following diagram commutes

$$\begin{array}{ccccccccc} 0 & \longrightarrow & E_1 & \xrightarrow{\varphi} & E & \xrightarrow{\psi} & E_2 & \longrightarrow & 0 \\ & & \| & & \downarrow \theta & & \| & & \\ 0 & \longrightarrow & E_1 & \xrightarrow[\varphi']{} & E' & \xrightarrow[\psi']{} & E_2 & \longrightarrow & 0 \end{array}$$

the extension $0 \to E_1 \xrightarrow{\varphi} E \xrightarrow{\psi} E_2 \to 0$ is trivial if $E \cong E_1 \oplus E_2$

& $\phi : E_1 \longrightarrow E$ canonical embedding & $\psi : E \to E_2$ canonical projection
$e_1 \longmapsto (e_1, e_2)$ $\qquad (e_1, e_2) \longmapsto e_2$

Prop the isom. classes of extensions of E_2 by E_1 are one-to-one correspondence with the elements of group

$$H^1(X, \mathcal{H}om(E_2, E_1)) = H^1(X, E_2^\vee \otimes E_1)$$

with trivial extensions of E_2 by E_1 $\longleftrightarrow$ zero element of $H^1(X, E_2^\vee \otimes E_1)$

<u>sketch of proof</u>. For an extension of E_2 by E_1

$$0 \to E_1 \to E \to E_2 \to 0$$

$\rightsquigarrow$ another exact seq.

$$0 \to \mathcal{H}om(E_2, E_1) \to \mathcal{H}om(E_2, E) \longrightarrow \mathcal{H}om(E_2, E_2) \to 0$$

taking cohomology, we obtain a canonical boundary map

$$\delta : \underset{\displaystyle \mathrm{Hom}(E_2, E_2)}{\overset{\displaystyle H^0(X, \mathcal{H}om(E_2, E_2))}{\|}} \longrightarrow H^1(X, \mathcal{H}om(E_2, E_1))$$

$$id_{E_2} \longmapsto \delta(id_{E_2}) \in H^1(X, \mathcal{H}om(E_2, E_1))$$

called the class of the extension $0 \to E_1 \to E \to E_2 \to 0$

$\square$

As an end of this section, we give a special case of Grothendieck's theorem on splitting of vector bundles on $\mathbb{P}^1$

THEOREM

$\forall$ locally free $\mathcal{O}_{\mathbb{P}^1}$-module of rk 2 is isomorphic to $\mathcal{O}_{\mathbb{P}^1}(m) \oplus \mathcal{O}_{\mathbb{P}^1}(n)$ for suitable integers $m, n \in \mathbb{Z}$.

Pf. Let E be a locally free sheaf of rank 2 on $\mathbb{P}^1$ & M be any invertible $\mathcal{O}_{\mathbb{P}^1}$-module.

Recall that for $\forall$ locally free sheaf $\mathcal{F}$ of rank n, the pairing

$$\Lambda^r \mathcal{F} \otimes \Lambda^{n-r}\mathcal{F} \longrightarrow \Lambda^n \mathcal{F}$$ is perfect for any r,

that is, $\Lambda^r \mathcal{F} \cong (\Lambda^{n-r}\mathcal{F})^\vee \otimes (\Lambda^n \mathcal{F})$

$$\Rightarrow \quad E \cong E^\vee \otimes \Lambda^2 E$$

$$\deg(E\otimes M) = \deg\left(\Lambda^2(E\otimes M)\right) = \deg E + 2\deg M$$

Choose M as follows:

$$M = \begin{cases} \mathcal{O}_{\mathbb{P}^1}(-\frac{d}{2}), & \text{if } d = \deg E \text{ even.} \\ \mathcal{O}_{\mathbb{P}^1}(-\frac{d+1}{2}), & \text{if } d = \deg E \text{ odd.} \end{cases}$$

& put $E' = E\otimes M$, then

$$\deg E' = \deg E + 2\deg M = 0 \text{ or } -1$$

Suffices to prove the theorem for E' & so WMA

$d = \deg E = 0$ or -1.

By Riemann-Roch,

$$\chi(E) = \deg E + \operatorname{rank} E\,(1 - g(C))$$
$$= d + 2 \geq 1$$

$$\Rightarrow \quad h^0(E) \geq 1$$

let $0 \neq s \in H^0(E)$, then s is a global section of the invertible subsheaf $L_s := i_s^* \mathcal{O}_{\mathbb{P}(E^\vee)}(-1)$. $\Rightarrow \deg L_s \geq 0$.

Moreover, the quotient $L' := E/L_s$ is also invertible.

$\left.\begin{array}{l} L_s, L' \text{ line bundles on } \mathbb{P}^1 \\ \operatorname{Pic}(\mathbb{P}^1) \cong \mathbb{Z} \end{array}\right\} \Rightarrow$ $\exists$ two integers $a, b \in \mathbb{Z}$

s.t. $L_s \cong \mathcal{O}_{\mathbb{P}^1}(a)$, with $\underline{a \geq 0}$ (since $\deg L_s \geq 0$)

$L' \cong \mathcal{O}_{\mathbb{P}^1}(b)$

$$\underset{\substack{\| \\ d}}{\deg E} = \deg L_s + \deg L' = a + b$$

we have a.s.e.s.

$$0 \to \mathcal{O}_{\mathbb{P}^1}(a) \longrightarrow E \longrightarrow \mathcal{O}_{\mathbb{P}^1}(d-a) \longrightarrow 0$$

with $d = \deg E = 0$ or -1.

$\Rightarrow$ E is an extension of $\mathcal{O}_{\mathbb{P}^1}(d-a)$ by $\mathcal{O}_{\mathbb{P}^1}(a)$.

By above prop,

$$H^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2a-d)) \cong \left\{ \begin{array}{l} \text{isom. classes of extensions of } \mathcal{O}_{\mathbb{P}^1}(d-a) \\ \text{by } \mathcal{O}_{\mathbb{P}^1}(a) \end{array} \right\}$$

$$\left.\begin{array}{l} a \geq 0 \\ d = 0 \text{ or } -1 \end{array}\right\} \Rightarrow 2a-d \geq -1. \Rightarrow h^1(\mathbb{P}^1, \mathcal{O}_{\mathbb{P}^1}(2a-d)) = 0$$

$$\Downarrow$$

the extension

$$0 \to \mathcal{O}_{\mathbb{P}^1}(a) \to E \to \mathcal{O}_{\mathbb{P}^1}(d-a) \to 0$$

$E \cong \mathcal{O}_{\mathbb{P}^1}(a) \oplus \mathcal{O}_{\mathbb{P}^1}(d-a) \Longleftarrow$ is trivial.

$\square$