Positivity of divisors

Setting : | X complete scheme $/\mathbb{C}$
| L line bundle on X

§1. Ample line bundles

Def 1 (very ample, ample line bundle)

- Call L is very ample if $\exists$ a closed embedding $X \overset{\iota}{\hookrightarrow} \mathbb{P}^N$ such that

$$L = \mathcal{O}_X(1) \overset{\text{def}}{=} \iota^*\mathcal{O}_{\mathbb{P}^N}(1) = \mathcal{O}_{\mathbb{P}^N}(1)|_X$$

- Call L is ample if $L^{\otimes m}$ is very ample for some $m>0$.

- a Cartier divisor $D \subset X$ is ample / very ample if the corresponding line bundle $\mathcal{O}_X(D)$ is so.

Example (ample line bundles on curves)

C : irreducible curve

L : line bundle on C

then L ample $\iff$ $\deg L > 0$.

Pf "$\Rightarrow$" If L ample, then $L^{\otimes m} = \mathcal{O}_{\mathbb{P}^N}(1)|_C$ for some $m>0$

$$\deg L > 0 \Leftarrow \deg \mathcal{O}_{\mathbb{P}^N}(1)|_C > 0$$

"$\Leftarrow$" If $\deg L > 0$, then for $n \gg 0$, $\deg L^{\otimes n} \geq 2g(C)+1$

$\Rightarrow$ $L^{\otimes n}$ very ample

$\Rightarrow$ L ample. $\square$

The amplitude/ampleness can be detected cohomologically.

THEOREM (Cartan-Serre-Grothendieck theorem)

L a line bundle on a complete scheme X. TFAE

① L ample

② (Serre vanishing) given $\forall$ coherent sheaf $\mathcal{F}$ on X, $\exists\ \overset{m_1(\mathcal{F})}{\overset{\|}{m_1}} \in \mathbb{Z}_{>0}$ such that the higher cohomology groups vanish

$$H^i(X, \mathcal{F}\otimes L^{\otimes m}) = 0 \quad \text{for } \forall i>0 \ \& \ \forall m \geq m_1.$$

③ given $\forall$ coherent sheaf $\mathcal{F}$ on X, $\exists\ m_2 \in \mathbb{Z}_{>0}$ s.t.

$\mathcal{F}\otimes L^{\otimes m}$ is generated by global sections for $\forall m \geq m_2$.

④ $\exists\ m_3 \in \mathbb{Z}_{>0}$ s.t. $L^{\otimes m}$ very ample for $\forall m \geq m_3$.

Sketch of proof

① $\Rightarrow$ ②

L ample $\Rightarrow$ $L^{\otimes m_0}$ very ample for some $m_0 > 0$

$\exists$ closed embedding $\iota : X \hookrightarrow \mathbb{P}^N$ s.t. $L^{\otimes m_0} = \mathcal{O}_X(1)$

Write $m = k m_0 + \ell$, $0 \le \ell \le m_0 - 1$

$$H^i(X, \mathcal{F} \otimes L^{\otimes m}) \cong H^i(\mathbb{P}^N, \iota_*(\mathcal{F} \otimes L^{\otimes \ell} \otimes \iota^* \mathcal{O}_{\mathbb{P}^N}(k))$$
$$\cong H^i(\mathbb{P}^N, \iota_*(\mathcal{F} \otimes L^{\otimes \ell}) \otimes \mathcal{O}_{\mathbb{P}^N}(k))$$
$$\| \text{ (Serre)}$$
$$0 \quad \cancel{\text{for } m \gg 0}.$$

which holds for each $0 \le \ell \le m_0 - 1$ & ~~$k \gg 0$~~ $k \ge k_0$ (given by Serre vanishing for cohomology on $\mathbb{P}^N$) Put $m_1 = (1+k_0) m_0$

② $\Rightarrow$ ③

Fix a point $x \in X$,

$$\mathfrak{m}_x \subset \mathcal{O}_{x,X} \quad \text{maximal ideal}$$

Consider the maximal ideal sheaf associated to this ideal

$$\mathfrak{m}_x(U) = \begin{cases} \mathcal{O}_X(U), & x \notin U \\ \{s \in \mathcal{O}_X(U) \mid s_x \in \mathfrak{m}_x\}, & x \in U \end{cases}$$

$i : \operatorname{Spec} k(x) \hookrightarrow X$

In fact, $\mathfrak{m}_x = \operatorname{kernel}(\mathcal{O}_X \longrightarrow i_* \mathcal{O}_{\operatorname{Spec} k(x)})$

Consider the exact seq.

$$0 \to \mathfrak{m}_x \cdot \mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/\mathfrak{m}_x \cdot \mathcal{F} \longrightarrow 0$$

twisted by $L^{\otimes m}$

$$0 \to (\mathfrak{m}_x \cdot \mathcal{F}) \otimes L^{\otimes m} \longrightarrow \mathcal{F} \otimes L^{\otimes m} \longrightarrow \mathcal{F}/\mathfrak{m}_x \cdot \mathcal{F} \otimes L^{\otimes m} \to 0$$

by ②, $\exists\, m_2 = m_2(\mathcal{F}, x)$ s.t. $H^1(\mathfrak{m}_x \cdot \mathcal{F} \otimes L^{\otimes m}) = 0$ for $m \ge m_2$

$$\Rightarrow \quad H^0(\mathcal{F} \otimes L^{\otimes m}) \twoheadrightarrow H^0(\mathcal{F}/\mathfrak{m}_x \cdot \mathcal{F} \otimes L^{\otimes m})$$

i.e. $\mathcal{F} \otimes L^{\otimes m}$ is globally generated in a neighborhood of x for $\forall m \ge m_2$

By quasi-compactness,

Can choose a uniform positive integer m_2 working for $\forall x \in X$.

③ $\Rightarrow$ ④

By ③, $\exists\, k_1 \in \mathbb{Z}_{>0}$ s.t. $L^{\otimes m}$ globally generated for $\forall\, m \ge k_1$

Consider the morphism defined by $|L^{\otimes m}|$

$$\varphi_m : X \longrightarrow \mathbb{P}^{h^0(L^{\otimes m}) - 1}$$

Claim: for $m \gg 0$, φ_m is a closed embedding.

Suffices to show φ_m is injective & unramified [II Prop 7.3]

To this end, Consider the set

$$U_m := \{ y \in X \mid L^{\otimes m} \otimes \mathfrak{m}_y \text{ is globally generated} \}$$

It is an open set & $U_m \subseteq U_{m+k}$ for $k \geq k_1$.

Given $\forall x \in X$,

by ③, can find integer $m_2(x)$ s.t. $x \in U_m$ for $m \geq m_2(x)$

$\Rightarrow X = \bigcup U_m$

By quasi-compactness $\exists$ a single integer $m_3 \geq k_1$ s.t.

$L^{\otimes m} \otimes \mathfrak{m}_x$ is generated by its global sections for $\forall x \in X$

whenever $m \geq m_3$.

$L^{\otimes m} \otimes \mathfrak{m}_x$ globally generated $\Rightarrow$ $\varphi_m(x) \neq \varphi_m(x')$ for $\forall x \neq x'$

φ_m unramified at x

$\Downarrow$

φ_m closed embedding for $\forall m \geq m_3$

④ $\Rightarrow$ ① by definition. $\square$

Proposition (ampleness under finite pullbacks)

$f: X \to Y$ finite morphism of complete schemes

L : ample line bundle on Y

then f^*L is an ample line bundle on X.

In particular, if $X \subseteq Y$ a subscheme, then $L|_X$ is ample.

Pf let $\mathcal{F}$ be a coherent sheaf on X.

$$f_*(\mathcal{F} \otimes f^* L^{\otimes m}) \underset{P.F}{\cong} f_* \mathcal{F} \otimes L^{\otimes m}$$

f finite $\Rightarrow$ $R^j f_*(\mathcal{F} \otimes f^* L^{\otimes m}) = 0$ for $\forall j > 0$.

$\Downarrow$

$$H^i(X, \mathcal{F} \otimes f^* L^{\otimes m}) = H^i(Y, f_*(\mathcal{F} \otimes f^* L^{\otimes m}))$$
$$\| $$
$$H^i(Y, f_* \mathcal{F} \otimes L^{\otimes m})$$

for each $i > 0$.

$\Downarrow$ Cartan-Serre-Grothendieck

f^*L ample on X $\square$

Prop (detecting ampleness reduced to reduced & irred. var.)

X complete scheme, L line bundle on X. Then

(1) L ample on $X \iff L$ ample on X_{red}

(2) L ample on $X \iff L|_{X_i}$ ample on X_i for each irreducible component $X_i \subseteq X$.

Pf (1) "$\Rightarrow$" $\left.\begin{array}{l} X_{red} \to X \text{ finite morphism} \\ L \text{ ample on } X \end{array}\right\} \Rightarrow L$ ample on X_{red}

"$\Leftarrow$" If L ample on X_{red}, we use the Serre vanishing to prove the ampleness of L on X.

Fix a coherent sheaf $\mathcal{F}$ on X, let $\mathcal{N}$ be the nilradical of $\mathcal{O}_X$, say $\mathcal{N}^r = 0$ for some $r \geq 0$.

Consider the filtration

$$\mathcal{F} \supset \mathcal{N}\cdot\mathcal{F} \supset \mathcal{N}^2\cdot\mathcal{F} \supset \cdots \supset \mathcal{N}^r\cdot\mathcal{F} = 0$$

the subquotients $\mathcal{N}^i\mathcal{F}/\mathcal{N}^{i+1}\mathcal{F}$ are coherent $\mathcal{O}_{X_{red}}$-modules.

$$\xRightarrow[\text{on } X_{red}]{L \text{ ample}} H^j(X_{red}, (\mathcal{N}^i\mathcal{F}/\mathcal{N}^{i+1}\mathcal{F}) \otimes L^{\otimes m}) = 0 \text{ for } j > 0 \text{ \& } m \gg 0$$

$$\| $$

$$H^j(X, (\mathcal{N}^i\mathcal{F}/\mathcal{N}^{i+1}\mathcal{F}) \otimes L^{\otimes m})$$

Consider exact sequences

$$0 \to \mathcal{N}^{i+1}\mathcal{F} \to \mathcal{N}^i\mathcal{F} \to \mathcal{N}^i\mathcal{F}/\mathcal{N}^{i+1}\mathcal{F} \to 0$$

twisted by $L^{\otimes m}$

$$0 \to \mathcal{N}^{i+1}\mathcal{F} \otimes L^{\otimes m} \to \mathcal{N}^i\mathcal{F} \otimes L^{\otimes m} \to \mathcal{N}^i\mathcal{F}/\mathcal{N}^{i+1}\mathcal{F} \otimes L^{\otimes m} \to 0$$

taking cohomology, we find by descending induction on i that

$$H^j(X, \mathcal{N}^i\mathcal{F} \otimes L^{\otimes m}) = 0 \text{ for } j > 0 \text{ \& } m \gg 0.$$

In particular, take $i = 0$, we obtain the desired vanishing result.

(2) "$\Rightarrow$" $\left.\begin{array}{l} X_i \hookrightarrow X \\ L \text{ ample on } X \end{array}\right\} \Rightarrow L$ ample on X_i

"$\Leftarrow$" Based on item (1), we may assume X reduced, say

$X = X_1 \cup \cdots \cup X_r$

By hypothesis, $L|_{X_i}$ ample for each i.

Fix a coherent sheaf $\mathcal{F}$ on X, & let I be the ideal sheaf of X_1 in X & Consider the s.e.s

$$0 \to I\cdot\mathcal{F} \longrightarrow \mathcal{F} \longrightarrow \mathcal{F}/I\cdot\mathcal{F} \longrightarrow 0$$

The outer terms

$I\cdot\mathcal{F}$ supported on $X_2 \cup \cdots \cup X_r$

$\mathcal{F}/I\cdot\mathcal{F}$ supported on X_1

So by induction on $\#\{\text{irred. components of } X\}$, WMA

$$H^j(X, I\mathcal{F}\otimes L^{\otimes m}) = H^j(X, (\mathcal{F}/I\mathcal{F})\otimes L^{\otimes m}) = 0$$

for $j>0$ & $m \gg 0$.

It then follows from above s.e.s. that

$$H^j(X, \mathcal{F}\otimes L^{\otimes m}) = 0 \quad \text{for } j>0 \ \& \ m\gg 0.$$

$\square$

Recall that Suppose X complete variety / scheme (over $\mathbb{C}$), L line bundle on X

$V \subset H^0(X, L)$ a non-zero finite-dim'l subspace

$|V| = \mathbb{P}(V)$ called a <u>linear system / series</u>

Evaluation of sections in V gives rise to a morph. of v.b.

$$ev : V \otimes_{\mathbb{C}} \mathcal{O}_X \longrightarrow L$$

<u>Def</u> the base ideal of $|V|$, $\mathfrak{b}(|V|) := \mathrm{Image}(V\otimes_{\mathbb{C}} L^\vee \to \mathcal{O}_X)$

the base locus of $|V|$, $\mathrm{Bs}|V| = \mathrm{Supp}(\mathcal{O}_X/\mathfrak{b}(|V|))$

↑ closed subset of X cut out by the base ideal.

<u>Def</u> We say the linear system is free (or base point-free) if its base locus is empty, i.e. $\mathfrak{b}(|V|) = \mathcal{O}_X$.

A divisor D or line bundle L is free if $|D|$ or $|L|$ is so.

We also say L is <u>generated by its global sections</u> / <u>globally generated</u> if $|L|$ free i.e. $H^0(L)\otimes\mathcal{O}_X \twoheadrightarrow L$

i.e. $|L|$ bpf

Prop (globally generated line bundles)

Suppose that L is globally generated. let

$$\varphi = \varphi_{|L|} : X \longrightarrow \mathbb{P}^{h^0(L)-1}$$

be the morphism defined by $|L|$.

then L is ample $\Leftrightarrow$ φ is a finite morphism

$\Leftrightarrow$ $L.C > 0$ for $\forall$ irred. curve $C \subset X$.

pf. · If φ finite, $\mathcal{O}_{\mathbb{P}^N}(1)$ ample $\Big\} \Rightarrow L = \varphi^* \mathcal{O}_{\mathbb{P}^N}(1)$ ample

$\Rightarrow L.C = \varphi^* \mathcal{O}_{\mathbb{P}^N}(1).C = \deg C > 0$

· If φ not finite.

then $\exists$ positive-dim'l subvar. $Z \subset X$ s.t. $\varphi(Z) = pt$

$L = \varphi^* \mathcal{O}_{\mathbb{P}^N}(1)$ $\Longrightarrow$ $L|_Z \cong \mathcal{O}_Z$ not ample $\Downarrow$ L not ample

for $\forall$ irred. curve $C \subseteq Z$ $\Longleftarrow$ $L.C = 0$ $\lightning$

The ampleness is characterized numerically:

THEOREM (Nakai–Moishezon–Kleiman criterion)

L line bundle on a projective scheme X

then L ample $\Leftrightarrow$ $\int_Y c_1(L)^{\dim Y} > 0$ for $\forall$ positive-dim'l irred. subvar. $Y \subseteq X$ (including irred. components of X)

$\int_Y c_1(L)^{\dim Y} = L^{\dim Y}.[Y]$

Cor (Ampleness under finite pullbacks)

$f: X \longrightarrow Y$ finite, surjective morph. of projective schemes

L line bundle on Y

If f^*L ample on X, then L ample on Y.

pf let $V \subseteq Y$ be an irred. subvariety.

f surjective. $\Rightarrow$ $\exists$ irred. var. $W \subseteq X$ mapping finitely onto V

(starting with $f^{-1}(V)$, construct W by taking irred. components & cutting down by general hyperplanes)

then $\int_W (f^*L)^{\dim W} \underset{\text{P.F}}{=} \deg(W/V) \cdot \int_V L^{\dim V} > 0 \overset{\text{NMK}}{\Longrightarrow} L$ ample on Y. $\square$

Nakai-Moishezon Criterion

THEOREM (Nakai-Moishezon Criterion)

> a divisor D on a surface S is ample if and only if
> $D^2 > 0$ & $DC > 0$ for all irreducible curves $C \subset S$.

Pf "$\Rightarrow$" If D ample then mD very ample for some $m>0$.

$$\varphi_{|mD|} : S \xhookrightarrow{|mD|} \mathbb{P}^N$$

$$\Rightarrow \quad \deg S = (mD)^2 = m^2 D^2 > 0$$

$$\deg C = \varphi^* \mathcal{O}_{\mathbb{P}^N}(1) . C = mD.C > 0$$

"$\Leftarrow$" If H is a very ample divisor, then H is represented by an irreducible curve $\Rightarrow DH > 0$ by hypothesis

&

$D^2 > 0$

$\Rightarrow \dim|nD| \underset{n \to +\infty}{=} +\infty$

$\Rightarrow$

$mD \geq 0$ for some $m > 0$

Replacing D by mD, we may assume D is effective.

So can be viewed as a curve $D \subset S$
(possibly singular, reducible & non-reduced)

Step 1: [the sheaf $\mathcal{O}_S(D) \otimes \mathcal{O}_D$ is ample on $\underset{\text{curve}}{D}$]

- Suffices to show $\mathcal{O}_S(D) \otimes \mathcal{O}_{D_{red}}$ ample on D_{red}.

If $D = \bigcup_{i=1}^{r} C_i$ (C_i irreducible curve)

- suffices to show $\mathcal{O}_S(D) \otimes \mathcal{O}_{C_i}$ ample on each C_i.

If $\nu_i : \widetilde{C}_i \to C_i$ normalization of C_i

- Suffices to show $\nu_i^* (\mathcal{O}_S(D) \otimes \mathcal{O}_{C_i})$ ample on $\widetilde{C}_i$

(since ν_i finite surjective)

$\deg \nu_i^* (\mathcal{O}_S(D) \otimes \mathcal{O}_{C_i}) = D.C_i > 0 \Rightarrow \nu_i^* (\mathcal{O}_S(D) \otimes \mathcal{O}_{C_i})$ ample on smooth curve $\widetilde{C}_i$

$\square$

Step 2: [$\mathcal{O}_S(D)^{\otimes n} = \mathcal{O}_S(nD)$ is globally generated by sections for $n \gg 0$

Consider the natural sequence

$$0 \to \mathcal{O}_S(-D) \to \mathcal{O}_S \to \mathcal{O}_D \to 0$$

twisted by $\mathcal{O}_S(nD)$,

$$0 \to \mathcal{O}_S((n-1)D) \to \mathcal{O}_S(nD) \to \mathcal{O}_S(nD)|_D \to 0$$

taking cohomology,

$$\begin{tikzcd} 0 \arrow[r] & H^0((n-1)D) \arrow[r] & H^0(nD) \arrow[r] & H^0(nD|_D) \arrow[r] & H^1((n-1)D) \arrow[d] \\ & & \cdots & \underset{\overset{\parallel}{0}}{H^1(nD|_D)} \arrow[l] & H^1(nD) \arrow[l] \end{tikzcd}$$

by Step 1, $\mathcal{O}_S(D) \otimes \mathcal{O}_D = \mathcal{O}_S(D)|_D$ ample on D

$$\Downarrow$$

$$H^1(nD|_D) = 0 \quad \text{for } n \gg 0$$

$\Rightarrow$ $\dim H^1((n-1)D) \geq \dim H^1(nD)$

both vector spaces are finite-dim'l

$$\Bigg\} \Rightarrow h^1((n-1)D) \overset{\parallel}{} h^1(nD) \quad \text{for } n \gg 0$$

$$\Downarrow$$

$$H^0(nD) \twoheadrightarrow H^0(nD|_D) \quad \text{for } n \gg 0. \quad (*)$$

by Step 1, $\mathcal{O}_S(D)|_D$ ample on D

$\Rightarrow$ $\mathcal{O}_S(nD)|_D$ generated by globally sections for $n \gg 0$.

$$\Downarrow (*)$$

these sections lift to global sections of $\mathcal{O}_S(nD)$

$$\begin{tikzcd} \mathcal{O}_S(nD) \arrow[r] \arrow[d] & \mathcal{O}_S(nD)|_D \arrow[d, shift right] \\ H^0(nD) \arrow[r, two heads] & H^0(nD|_D) \cancel{\otimes \mathcal{O}_D} \arrow[u, shift right] \end{tikzcd}$$

More precisely, can find global sections ~~s~~

$$t_1, \cdots, t_k \in H^0(nD|_D)$$

Such that for each $x \in D$, at least one of the t_i does not vanish at x.

Now lifting these sections to $t_1', \cdots, t_k' \in H^0(\mathcal{O}_S(nD))$

& adding a section $t_0' \in H^0(\mathcal{O}_S(nD))$ s.t. $\mathrm{div}(t_0') = nD$

Vanishing only along D

Then we get $(k+1)$ global sections in $H^0(S, \mathcal{O}_S(nD))$ s.t. for $\forall\, x \in S$, at least one of these sections does not vanish at x.

$\square$

Fix an n s.t. $\mathcal{O}_S(nD)$ globally generated. $\rightsquigarrow$ a morphism $\underset{\varphi}{\overset{\varphi_{|nD|}}{\|}} : S \to \mathbb{P}^N$

Step 3 :

[the morphism $\varphi : S \xrightarrow{|nD|} \mathbb{P}^N$ has finite fibres]

Otherwise, $\exists$ an irreducible curve $C \subset S$ with $\varphi(C) = pt$.

In this case, taking a hyperplane $\overset{\mathcal{O}(1)}{}$ in $\mathbb{P}^N$ not through this point,

then $\underset{nD.C}{\overset{\varphi^*\mathcal{O}(1).C}{\|}} = \cancel{\phi}$ $\quad \varphi^*\mathcal{O}(1) \cap C = \phi$

i.e. $\underset{nD.C}{\overset{\varphi^*\mathcal{O}(1).C}{\|}} = 0 \quad \lightning$

Step 4 : [Conclusion]

projective morphism + quasi-finite $\Rightarrow$ finite morphism

$\Rightarrow$ φ is actually a finite morphism.

$\Rightarrow$ $\underset{\mathcal{O}_S(nD)}{\overset{\varphi^*\mathcal{O}(1)}{\|}}$ is ample on S

$\Rightarrow$ D ample on S

$\square$

§2 Nef line bundles

~~Def~~ Set-up: | X complete variety / scheme
| L line bundle on X

Def · L is <u>numerically effective</u> / nef if $\int_C c_1(L) \geq 0$ (where $\int_C c_1(L) = L.[C]$)

for $\forall$ irreducible curve $C \subset X$.

· a Cartier divisor D on X is nef if $D.C \geq 0$

for $\forall$ curve $C \subset X$.

Rmk the terminology "nef" introduced by Miles Reid, it's an abbreviation for "numerically eventually ~~ref~~ free".

Formal properties of nefness:

① $f: X \to Y$ proper morphism

If L nef on Y, then f^*L nef on X

In particular, if $X \subseteq Y$ subscheme, then $L|_X$ nef on X.

② If $f: X \to Y$ surjective, proper morphism &

f^*L nef on X

then the line bundle L nef on Y.

③ L nef on $X \Leftrightarrow L$ nef on X_{red}

④ L nef on $X \Leftrightarrow L|_{X_i}$ nef on each irred. component $X_i \subseteq X$.

<u>THEOREM</u> (Kleiman's theorem)

| X complete variety / scheme

| If D nef $\mathbb{R}$-divisor on X, then $D^k.V \geq 0$ for $\forall$ irreducible subvariety $V \subseteq X$ of dim k.

| If L nef line bundle on X, then $\int_V c_1(L)^{\dim V} \geq 0$

[nef divisors "=" limits of ample divisors]

<u>Cor</u> | X projective var / scheme.

| D nef $\mathbb{R}$-div on X

| If H is any ample $\mathbb{R}$-div. on X, then $D + \varepsilon \cdot H$ ample for $\forall \varepsilon > 0$

Conversely, if D, H two divisors s.t. $D+\varepsilon H$ ample for $\forall$ sufficiently small $\varepsilon>0$

then D is nef.

<u>pf</u>. If $D+\varepsilon H$ ample for $\varepsilon>0$, then

$$(D+\varepsilon H).C>0 \text{ for } \forall \text{ irred. curve } C\subset X$$
$$\| $$
$$D\cdot C+\varepsilon H.C$$

taking limit $\varepsilon\to 0$, one has

$$D.C\geq 0 \Rightarrow D \text{ nef.}$$

Conversely, if D nef & H ample. Replacing εH by H, suffices to show $D+H$ ample. To this end,

[the main point is to verify that $D+H$ satisfies the Nakai-type inequalities : $(D+H)^{\dim V}\cdot V>0$ for $\forall$ positive-dim'l irred. subvar. $V\subseteq X$.
If $D+H \equiv_{num}$ a $\mathbb{Q}$-divisor, this $\Rightarrow$ $D+H$ ample
In the general case, approximating $D+H$ by $\mathbb{Q}$-divisors $\Rightarrow$ $D+H$ ample

So fix an irred. subvariety $V\subseteq X$ of dim $k>0$.

We infer that

$$(D+H)^k\cdot V=\sum_{\ell=0}^{k}\binom{k}{\ell}\left(H^\ell\cdot D^{k-\ell}\cdot V\right)$$

H ample $\mathbb{R}$-divisor $\Rightarrow$ H is a positive $\mathbb{R}$-linear combination of integral ample divisors.

$\Downarrow$

$H^\ell\cdot V$ represented by an effective $(k-\ell)$-cycle with $\mathbb{R}$-coefficients.

$\Downarrow$ Kleiman's thm on each comp. of the cycle

$$(D+H)^k\cdot V>0 \Leftarrow \begin{cases} D^{k-\ell}\cdot(H^\ell\cdot V)\geq 0 \\ H^k\cdot V>0 \text{ by ampleness of } H\end{cases}$$

for each V

$\Downarrow$

If $D+H$ is $\mathbb{Q}$-divisor, then $D+H$ is ample.

It remains to prove if $D+H$ is irrational, then $D+H$ is ample.

To this end, choose ample divisors $H_1,\cdots,H_r$ whose classes span $N_1(X)_{\mathbb{R}}$ (Space of 1-cycles on X with $\mathbb{R}$-coefficients $/\equiv_{num}$)
$\uparrow$ $\dim=\rho(X)$ i.e. all finite $\mathbb{R}$-linear combinations of irred. curves on X modulo $\equiv_{num}$

By the open nature of ampleness, the $\mathbb{R}$-divisor

$$H(\varepsilon_1, \cdots, \varepsilon_r) = H - \varepsilon_1 H_1 - \cdots - \varepsilon_r H_r$$

is ample for all $0 \leq \varepsilon_i << 1$.

$$\{ H(\varepsilon_1, \cdots, \varepsilon_r) \mid 0 \leq \varepsilon_i << 1 \} \overset{\text{open}}{\subset} N^1(X)_{\mathbb{R}}$$

w.r.t. usual topology on the f.d. $\mathbb{R}$-vector space $N^1(X)_{\mathbb{R}}$

$\Rightarrow \exists\ 0 < \varepsilon_i << 1$ s.t. $D' = D + H(\varepsilon_1, \cdots, \varepsilon_r)$ represents a <u>rat'l class</u> in $N^1(X)_{\mathbb{R}}$.

$\Downarrow$

D' ample

$\Downarrow$

$$\underbrace{D' + \varepsilon_1 H_1 + \cdots + \varepsilon_r H_r}_{= D + H} \text{ ample}$$

$\square$

<u>Cor</u>

X projective variety / scheme

H ample $\mathbb{R}$-divisor on X

D a fixed $\mathbb{R}$-divisor on X

then D is ample $\Leftrightarrow \exists\ \varepsilon > 0$ s.t. $\dfrac{D.C}{H.C} \geqslant \varepsilon$ for $\forall$ irreducible curve $C \subseteq X$.

(i.e. the ampleness of a divisor is characterized by the uniformly bounded below property on its degree on any curve bounded below in terms of ~~its the curve~~ the degree of the curve w.r.t. a known ample div.)

<u>Pf.</u> $\dfrac{D.C}{H.C} \geqslant \varepsilon$ for $\forall$ curve $C \Leftrightarrow D - \varepsilon H$ is nef

If $D - \varepsilon H$ nef, then $\underbrace{(D - \varepsilon H)}_{\text{nef}} + \underbrace{\varepsilon H}_{\text{ample}} = D$ ample.

If D ample, then by the open nature of ampleness, $D - \varepsilon H$ is ample for $0 \leq \varepsilon \ll 1$. In particular, $D - \varepsilon H$ nef.

$\square$

THEOREM (Seshadri's criterion)

X projective var.

D divisor on X

then D ample $\iff \exists\ \varepsilon > 0$ s.t. $\dfrac{D.C}{\mathrm{mult}_x C} \geq \varepsilon$

for $\forall\, x \in X$ & $\forall$ irred. curve $C \subseteq X$ through x.

(i.e. degree of any curve is uniformly bounded below in terms of its sing. $\Updownarrow$ ampleness)

Pf. "$\Rightarrow$" If E_x is an effective divisor through x & meeting an irred. curve C properly, then

$$\mathrm{mult}_x(E_x \cap C) \geq \mathrm{mult}_x C$$

In particular $E_x . C \geq \mathrm{mult}_x C$

If D ample $\Rightarrow$ mD very ample for some $m \gg 0$.

$\Rightarrow$ for $\forall\, x$ & C, one can find an effective divisor $E_x \underset{\text{lin}}{\sim} mD$ with above stated properties.

$\Rightarrow$ $D.C \geq \frac{1}{m}\mathrm{mult}_x(C)$ for $\forall\, x$ & C.

"$\Leftarrow$" Arguing by induction on $\dim X = n$

We can assume that $\mathcal{O}_V(D)$ ample for $\forall$ irred. proper subvar. $V \subset X$

In particular, $D^{\dim V}. V > 0$ for $\forall$ proper $V \subset X$ of positive dim.

By Nakai-Moishezon-Kleiman Criterion, suffices to show $D^n > 0$.

To this end, fix any smooth point $x \in X$ & consider the blow-up of X at x with except'l div. $E = \mu^{-1}(x)$.

$$\mu : X' = \mathrm{Bl}_x(X) \longrightarrow X$$

claim: the $\mathbb{R}$-divisor $\mu^* D - \varepsilon.E$ is nef on X'.

Granting this claim for the moment,

$$\text{Kleiman's theorem} \Rightarrow \underbrace{(\mu^* D - \varepsilon.E)^n}_{= D^n - \varepsilon} \geq 0$$

$\Rightarrow D^n > 0$.

For the ~~neff~~ nefness of $\mu^* D - \varepsilon E$.

Fix an irreducible curve $C' \subset X'$ not contained in E

& $C := \mu(C')$ s.t. C' is the proper transform of C.

$\Longrightarrow \quad C'.E = \mathrm{mult}_x C$

On the other hand,

$$\mu^* D . C' \underset{\text{P.F.}}{=} D.C \overset{\text{(by hypothesis)}}{\geqslant} \varepsilon \cdot \mathrm{mult}_x C$$

$$\| $$

$$\varepsilon \cdot EC'$$

$$\Downarrow$$

$$(\mu^* D - \varepsilon E) C' \geqslant 0 . \quad (*)$$

Since $\mathcal{O}_E(E)$ is a negative line bundle on the projective space E

$(*)$ holds for $\underset{\text{curve}}{C'} \subset E$.

$\Longrightarrow \quad \mu^* D - \varepsilon E$ is nef.

$\square$

§3 Big line bundles

X : irreducible projective variety $/\mathbb{C}$

L: line bundle on X

If for some ~~non-negative~~ positive integer $m \geq 0$, $H^0(X, L^{\otimes m}) \neq 0$

Consider the rational map associated to $|L^{\otimes m}|$

$$\varphi_m = \varphi_{|L^{\otimes m}|} : X \dashrightarrow \mathbb{P}^{h^0(L^{\otimes m})-1}$$

[Iitaka dim]

Def • Assume X is normal, define the Iitaka dimension of L to be

$$\underset{\|}{\kappa(L)} := \max_{\substack{H^0(L^{\otimes m}) \neq 0 \\ \cancel{m \geq 0} \\ m > 0}} \{ \dim \varphi_m(X) \}$$
$$\kappa(X, L)$$

If for all $m > 0$, $H^0(X, L^{\otimes m}) = 0$, define

$$\kappa(X, L) := -\infty .$$

• For non-normal variety X, pass to its normalization $\nu : X' \longrightarrow X$ & put

$$\kappa(X, L) := \kappa(X', \nu^* L)$$

• for a Cartier divisor D on X, set

$$\kappa(X, D) := \kappa(X, \mathcal{O}_X(D)) .$$

$\boxed{\text{Summary}}$ One has either $\kappa(X, L) = -\infty$

or $0 \leq \kappa(X, L) \leq \dim X$

Def [Kodaira dim]

• X smooth projective variety, K_X its canonical divisor.

$\underset{\underset{\kappa(X)}{\|}}{\kappa(X, K_X)}$ called the Kodaira dimension of X

• X singular var.

$\kappa(X) := \kappa(X')$ where X' is any smooth model of X

Kodaira dimension is the most basic birational invariant of a var.

Semi-ample line bundles

Def a line bundle L on a complete scheme is <u>semiample</u> if $L^{\otimes m}$ is globally generated for some $m>0$.

a divisor D is semiample if $\mathcal{O}_X(D)$ is so.

Def an <u>algebraic fibre space</u> is a surjective projective morphism $f: X \to Y$ of reduced, irred. var. such that $f_*\mathcal{O}_X = \mathcal{O}_Y$

(f has connected fibres)

lemma (pullbacks via a fibre space)

$f: X \to Y$ an algebraic fibre space

L line bundle on Y

then $H^0(X, f^*L^{\otimes m}) = H^0(Y, L^{\otimes m})$ for $\forall m \geq 0$

In particular, $\kappa(X, f^*L) = \kappa(Y, L)$

pf $H^0(X, f^*L^{\otimes m}) = H^0(Y, f_*(f^*L^{\otimes m})) \overset{\text{P.F}}{=} H^0(Y, f_*\mathcal{O}_X \otimes L^{\otimes m})$

$\| \leftarrow f$ alg fibre sp

$H^0(Y, L^{\otimes m})$ □

lemma (injectivity of Picard groups)

X, Y irred. projective var.

$f: X \to Y$ algebraic fibre space

then $f^*: \operatorname{Pic} Y \to \operatorname{Pic} X$ is injective

pf. Suppose L a line bundle on Y s.t. $f^*L \cong \mathcal{O}_X$.

then $H^0(Y, L) = H^0(X, f^*L) = H^0(X, \mathcal{O}_X) \neq 0$

$H^0(Y, L^\vee) = H^0(X, f^*L^\vee) = H^0(X, \mathcal{O}_X) \neq 0$.

$\Longrightarrow L \cong \mathcal{O}_Y$.

lemma (normality of fibre spaces)

$f: X \to Y$ algebraic fibre space

If X normal, then Y is also normal

pf let $\nu: Y' \to Y$ be the normalization of Y.

then f factors through ν.

f fibre space $\Rightarrow$ ν must be an isom.

□

[$\mathbb{Q}$-divisors] $\mathbb{Q}$-divisors & $\mathbb{R}$-divisors

Def ($\mathbb{Q}$-divisors)

X alg var./scheme, a **$\mathbb{Q}$-divisor** on X is an element of the $\mathbb{Q}$-vector space $\mathrm{Div}_{\mathbb{Q}}(X) := \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{Q}$

represent a $\mathbb{Q}$-div. $D \in \mathrm{Div}_{\mathbb{Q}}(X)$ as a finite sum

$$D = \sum c_i D_i \quad \text{with } c_i \in \mathbb{Q},\ D_i \in \mathrm{Div}(X)$$

integral Cartier div.

by clearing dominators, can write

$$D = cA$$

for a single rat'l number $c \in \mathbb{Q}$ & an integral divisor A

If $c \neq 0$, then $cA = 0 \Leftrightarrow A$ is a torsion element of $\mathrm{Div}(X)$.

a $\mathbb{Q}$-divisor D is **integral** if it is in the image of the natural map $\mathrm{Div}(X) \to \mathrm{Div}_{\mathbb{Q}}(X)$

The $\mathbb{Q}$-divisor D is **effective** if it is of the form

$$D = \sum c_i D_i$$

with $c_i \geq 0$ & D_i effective.

Def (Support)

$D \in \mathrm{Div}_{\mathbb{Q}}(X)$ a $\mathbb{Q}$-divisor

Say a codim 1 subset $E \subseteq X$ is a support of D if D admits a representation $D = \sum c_i D_i$ with

$$\bigcup_i \mathrm{Supp}\, D_i \subseteq E.$$

Def (equivalences & operations on $\mathbb{Q}$-div.)

X complete var./scheme.

① given a subvar/subscheme $V \subseteq X$ of pure dim k.

a $\mathbb{Q}$-valued intersection product

$$\mathrm{Div}_{\mathbb{Q}}(X) \times \cdots \times \mathrm{Div}_{\mathbb{Q}}(X) \longrightarrow \mathbb{Q}$$

$$(D_1, \cdots, D_k) \longmapsto \int_V D_1 \cdots D_k = D_1 \cdots D_k \cdot [V]$$

induced from the intersection prod. on $\mathrm{Div}(X)$.

② two $\mathbb{Q}$-div. D_1, D_2 are **numerically equivalent**, $D_1 \equiv_{\mathrm{num}, \mathbb{Q}} D_2$

if $D_1 C = D_2 C$ for $\forall$ curve $C \subseteq X$.

$$N^1(X)_{\mathbb{Q}} := \mathrm{Div}_{\mathbb{Q}}(X) / \equiv_{\mathrm{num}}.$$

③ two $\mathbb{Q}$-div. D_1, D_2 are linearly equivalent, $D_1 \underset{lin,\mathbb{Q}}{\sim} D_2$

if $\exists$ integer m s.t. mD_1 & mD_2 integral & linearly equivalent as integral Cartier divisors.

④ (pullbacks)

let $f: X \longrightarrow Y$ morph. s.t. the image of every associated subvar. of X meets a support of $D \in \mathrm{Div}_{\mathbb{Q}}(Y)$ properly.

then $f^*D \in \mathrm{Div}_{\mathbb{Q}}(X)$ induced from pullbacks on integral div.

$$\rightsquigarrow \quad f^*: N^1(Y)_{\mathbb{Q}} \longrightarrow N^1(X)_{\mathbb{Q}}.$$

Def (ampleness for $\mathbb{Q}$-div.)

$D \in \mathrm{Div}_{\mathbb{Q}}(X)$ a $\mathbb{Q}$-divisor.

D is <u>ample</u> if any one of the following 3 equivalent conditions holds:

① D is of the form $D = \sum c_i D_i$ where $\begin{cases} c_i \in \mathbb{Q}_{>0} \\ D_i \text{ ample Cartier div.} \end{cases}$

② $\exists\, m \in \mathbb{Z}_{>0}$ s.t. mD integral & ample

③ D satisfies Nakai-type ineq:

$$(D^{\dim V} \cdot V) > 0 \text{ for } \forall \text{ irred. subvar. } V \subseteq X \text{ of positive dim}$$

Prop (ampleness is an open condition)

X projective var, H ample $\mathbb{Q}$-div. on X

E arbitrary $\mathbb{Q}$-divisor.

then $H + \varepsilon E$ ample for $\forall$ suff. small rat'l numbers $0 \leq |\varepsilon| \ll 1$

More generally, given finitely many $\mathbb{Q}$-div. $E_1, \cdots, E_r$ on X

$H + \varepsilon_1 E_1 + \cdots + \varepsilon_r E_r$ is ample for all suff. small rational numbers $0 \leq |\varepsilon_i| \ll 1$.

pf. Clearing denominators. WMA H & each E_i are integral.

by taking $m \gg 0$, we can ~~arrange for~~ arrange that

$$\underbrace{mH \pm E_1, \ \cdots, \ mH \pm E_r}_{2r \text{ divisors}} \text{ are ample.}$$

Now if $|\varepsilon_i| \ll 1$, can write $H + \varepsilon_1 E_1 + \cdots + \varepsilon_r E_r$ as a positive $\mathbb{Q}$-linear combination of H & some of the $\mathbb{Q}$-div. $H \pm \frac{1}{m} E_i$.

$\Downarrow$ positive $\mathbb{Q}$-linear. combination of ample $\mathbb{Q}$-div.

$H + \varepsilon_1 E_1 + \cdots + \varepsilon_r E_r$ ample $\square$

[$\mathbb{R}$-divisors]

$\mathrm{Div}_{\mathbb{R}}(X) := \mathrm{Div}(X) \otimes_{\mathbb{Z}} \mathbb{R}$

$\uparrow$

$\mathbb{R}$-vector space of $\mathbb{R}$-divisors on X

If $D \in \mathrm{Div}_{\mathbb{R}}(X)$, can write $D = \sum c_i D_i$ with $c_i \in \mathbb{R}$, $D_i \in \mathrm{Div}(X)$

$D \equiv_{num,\mathbb{R}} 0$, if $D.C = 0$ for $\forall$ curve $C \subset X$.

$N^1(X)_{\mathbb{R}} := \mathrm{Div}_{\mathbb{R}}(X)/\equiv_{num}$.

D is effective, if $D = \sum c_i D_i$ with $c_i \geq 0$ & D_i effective

Def (ampleness for $\mathbb{R}$-div.)

X complete scheme / var.

an $\mathbb{R}$-div. D on X is <u>ample</u> if $D = \sum c_i D_i$

with $c_i > 0$, $c_i \in \mathbb{R}$ & D_i ample Cartier divisors.

As before, ampleness depends only on $\equiv_{num}$ classes.

Prop (Openness of ampleness for $\mathbb{R}$-div.)

X projective var, H ample $\mathbb{R}$-div. on X.

given finitely many $\mathbb{R}$-div. $E_1, \cdots, E_r$, then the $\mathbb{R}$-div

$H + \varepsilon_1 E_1 + \cdots + \varepsilon_r E_r$

is ample for $\forall$ suff. small real numbers $0 \leq |\varepsilon_i| \ll 1$.

Pf. • When H & each E_i are rat'l, it's true by openness nature of ampleness for $\mathbb{Q}$-div.

• each E_ℓ is a finite $\mathbb{R}$-linear combination of integral div.

WLOG assume all the E_ℓ are integral.

Write $H = \sum c_i D_i$ with $c_i \in \mathbb{R}_{>0}$ & D_i integral, ample.

Fix a rational number $0 < c < c_1$, then

$$H + \sum \varepsilon_\ell E_\ell = \Big(c D_1 + \sum \varepsilon_\ell E_\ell\Big) + (c_1 - c) D_1 + \sum_{i \geq 2} c_i D_i.$$

$\uparrow$ by the case already treated it is ample.

$\uparrow$ ample

$\uparrow$ ample

$\square$

Big line bundles & divisors

Def a line bundle L on an irreducible projective var. X is big if $\kappa(X, L) = \dim X$

a Cartier divisor $D \subset X$ is big if $\mathcal{O}_X(D)$ is so.

Def a variety X is said to be of general type if
sm proj

$\kappa(X) = \dim X$

$\Leftrightarrow$ K_X is big

$\Leftrightarrow$ for $d \gg 0$, $|dK_X|$ defines a birational map onto its image
(a priori)

Lemma X : projective variety of dim n

D a divisor on X

then D is big $\Leftrightarrow$ $\exists$ a constant $c > 0$ s.t.

$$h^0(X, \mathcal{O}_X(mD)) \geq c \cdot m^n$$

for all suff. large m s.t. $h^0(mD) \neq 0$

Def (Section ring of a line bundle)

L : line bundle on a projective var. X

the graded ring / **section ring** associated to L is the graded $\mathbb{C}$-algebra

$$\begin{array}{c} R(L) := \bigoplus_{m \geq 0} H^0(X, L^{\otimes m}) \\ \| \\ R(X, L) \end{array}$$

D Cartier divisor on X, $R(D) := R(\mathcal{O}_X(D))$.

Say a line bundle L is **finitely generated** if its section ring $R(X, L)$ is a finitely generated $\mathbb{C}$-alg.

THEOREM (Zariski)

L semi-ample line bundle on a normal projective var. X

then L is finitely generated.

Theorem (theorem of Zariski-Fujita)

L : line bundle on a projective variety X

If $Bs|L|$ is a finite set, then L is semi-ample, that is $L^{\otimes m}$ is free for some $m > 0$

Hodge index theorem

S : nonsingular projective surface $/ k=\bar{k}$

lemma 1 | D_1, D_2 two divisors on S

$E_2 \in |D_2|$

then the map $|D_1| \longrightarrow |D_1+D_2|$

$E \longmapsto E+E_2$

is injective.

In particular, $\dim|D_1| \leq \dim|D_1+D_2|$.

Pf. an easy consequence of the short exact sequence

$$0 \to \mathcal{O}_S(E) \to \mathcal{O}_S(E+E_2) \to \mathcal{O}_{E_2}(E+E_2) \to 0$$

lemma 2 D: a divisor on S with $D^2>0$

If $D^2>0$, then either nD or $-nD$ is linearly eq. to a nonzero effective div. for $n \gg 0$

H: a hyperplane section of S

then exactly one of the following two statements holds:

① $D.H>0$ & $\dim_{n\to+\infty}|nD| = +\infty$,

or

② $D.H<0$ & $\dim_{n\to-\infty}|nD| = +\infty$.

Pf. by Riemann-Roch,

$$\underset{h^0(nD)-h^1(nD)+h^0(K_S-nD)}{\underset{\|}{\chi(nD)}} = \chi(\mathcal{O}_S) + \underset{\frac{1}{2}n^2(D^2)-\frac{1}{2}nDK_S+\chi(\mathcal{O}_S)}{\underset{\|}{\frac{1}{2}\left((nD)^2-nD.K_S\right)}}$$

$$\Rightarrow \dim|nD| + \dim|K_S-nD| \geq \underset{\substack{\downarrow \\ +\infty \ (\text{as } n\to\pm\infty)}}{\frac{1}{2}n^2D^2-\frac{1}{2}nDK_S+\chi(\mathcal{O}_S)-2}$$

Claim 1 : | Cannot have both $\dim|nD|\to+\infty$ & $\dim|K_S-nD|\to+\infty$ as $n\to\pm\infty$

Indeed, otherwise for $n \gg 0$ (or $n \ll 0$), $\exists\ E \in |nD|$

by lemma 1, $\underset{\substack{\downarrow \\ +\infty}}{\dim|K_S-nD|} \leq \underset{\substack{\| \\ p_g(S)-1}}{\dim|K_S|}$ $\lightning$

□

Claim 2 | $\dim|nD|$ cannot tend to $+\infty$ for both $n\to+\infty$ and $n\to-\infty$

Indeed, if $|nD| \neq \phi$ for some $n>0$, then $HD \overset{H \text{ ample}}{>} 0$.

Then $|nD|=\phi$ for all $n<0$. (otherwise we would have $HD<0$ $\lightning$)

□

Claim 3 | $\dim|K_S - nD|$ cannot tend to $+\infty$, for both $n \to +\infty$ and $n \to -\infty$.

Indeed, otherwise for $n \gg 0$, $\exists\, E \in |K_S - nD|$

By lemma 1,

$$\underbrace{\dim|K_S + nD + K_S - nD|}_{= \dim|2K_S|} \geq \dim|K_S + nD| \to +\infty \ (\text{as } n \to -\infty) \ \lightning \quad \square$$

The lemma 2 follows from above three claims. $\square$

Corollary 3

| D : divisor on S
| H : a hyperplane section on S s.t. $D.H = 0$
| then $D^2 \leq 0$, and $D^2 = 0 \Leftrightarrow D \underset{num}{\equiv} 0$.

Pf. • if $D^2 > 0$, by lemma 2, either $HD > 0$ or $HD < 0$ $\lightning$

• if D not numerically equivalent to zero, then $\exists$ divisor $E \subseteq S$ s.t. $D.E \neq 0$.

Put $\tilde{E} := (H^2)E - (HE)H$, then

$$\begin{cases} \tilde{E}D = H^2.(ED) - (HE)\underbrace{(HD)}_{=0} = H^2(ED) \neq 0 \\ \tilde{E}H = H^2(HE) - (HE)H^2 = 0 \end{cases}$$

Now consider $\tilde{D} := \tilde{E} + nD$, we have

$$\begin{cases} H\tilde{D} = H\tilde{E} + nHD = 0 \\ \tilde{D}^2 = (\tilde{E} + nD)^2 = \tilde{E}^2 + 2n\tilde{E}D + n^2\underbrace{D^2}_{=0} = \tilde{E}^2 + 2n\tilde{E}D \end{cases}$$

if $\tilde{E}D > 0$, taking $n \gg 0$
(resp. $\tilde{E}D < 0$) (resp. $n \ll 0$)

We get $\tilde{D}^2 > 0$ & $H\tilde{D} = 0$ $\lightning$ (contradict to Lemma 2). $\square$

(Hodge Index Theorem)

| D, E two divisors on surface S
| $D^2 > 0$, $DE = 0$
| then $E^2 \leq 0$, and $E^2 = 0 \Leftrightarrow E \underset{num}{\equiv} 0$.

Pf. · $NS(S)_{\mathbb{R}} = NS(S)\otimes_{\mathbb{Z}}\mathbb{R}$ is a ρ-dim'l $\mathbb{R}$-vector space by Néron-Severi theorem, here $\rho = \rho(S)$ is the Picard #

Clearly, the intersection pairing $(-.-)$ is a non-degenerate symmetric bilinear form on $NS(S)_{\mathbb{R}}$.

· denote by h the class in $NS(S)_{\mathbb{R}}$ of a hyperplane section of S.

We complete h to a $\mathbb{R}$-basis of $NS(S)_{\mathbb{R}}$, say $h_1 = h, h_2, \cdots, h_\rho$ such that $h.h_i = 0$ for $i \geq 2$.

(this can be done, since if $hh_2 \neq 0$, taking $h_2' = \frac{hh_2}{h^2}h - h_2$ then $hh_2' = 0$ & $\{h, h_2\}$ and $\{h, h_2'\}$ are $\mathbb{R}$-linearly equivalent)

By Corollary 3, the intersection pairing $(-.-)$ has signature $(\underset{+1}{1}, \underset{-1}{\rho-1})$ in this basis.

Then the desired result follows from Sylvester's theorem on the invariance of signature of symmetric bilinear forms on $NS(S)_{\mathbb{R}}$.

(Weak form of Kleiman criterion)

If D is a nef divisor, then $\begin{cases} D^2 \geq 0 \ \& \\ D + \varepsilon H \text{ is ample for } \forall \varepsilon \in \mathbb{Q} \end{cases}$

Pf Consider the quadratic polynomial

$$P(t) = (D + tH)^2$$

then $P(t)$ is a continuous increasing function of $t \in \mathbb{Q}_{>0}$

&

$$P(t) = t^2H^2 + 2tHD + D^2 > 0 \text{ for } t \gg 0.$$

Claim: let $t \in \mathbb{Q}_{>0}$, then $P(t) > 0 \Rightarrow P(\frac{t}{2}) > 0$.

Indeed, $\left.\begin{matrix}(D+tH)^2 > 0 \\ H(D+tH) > 0\end{matrix}\right\} \Rightarrow n(D+tH) \underset{\text{lin.}}{\sim}$ effective divisor for suitable $n \gg 0$.

· Then D nef $\Rightarrow D(D+tH) \geq 0$

$$\Downarrow$$

$$(D + \tfrac{t}{2}H)^2 = D(D+tH) + \left(\tfrac{t}{2}\right)^2 H^2 > 0$$

by above claim, $D^2 = \lim\limits_{n \to +\infty} P(\frac{t}{2^n}) \geq 0$

For the 2nd statement, by Nakai-Moishezon criterion, Suffices to check that $\begin{cases} (D+\varepsilon H)^2 > 0 \\ (D+\varepsilon H).C > 0 \text{ for } \forall \text{ irred. curve } C \subset S. \end{cases}$

- D nef, H ample, $\varepsilon \in \mathbb{Q}_{>0} \Rightarrow (D+\varepsilon H)C > 0$ for $\forall$ curve C
- for $\forall\ \varepsilon \in \mathbb{Q}_{>0}$, $\exists\ m \in \mathbb{Z}_{>0}$ s.t. $\varepsilon > \frac{t_0}{2^m}$ here $p(t_0) > 0$

  then $(D+\varepsilon H)^2 > 0$ follows from the fact that $P(t)$ is a continuous increasing function of $t \in \mathbb{Q}$ for $t > 0$. & above claim.

$\square$

Corollary 4

D, E two divisors on surface S

$D^2 \geq 0$

then $D^2E^2 \leq (DE)^2$ i.e. $\begin{vmatrix} D^2 & DE \\ DE & E^2 \end{vmatrix} \leq 0$

and the equality holds $\Leftrightarrow \exists\ r \in \mathbb{Q}$, s.t. $E \underset{num}{\equiv} rD$.

Pf If $D^2 = 0$, there is nothing to prove

If $D^2 > 0$, note that

$$D\big((DE)D - (D^2)E\big) = 0$$

by Hodge index theorem, $\big((DE)D - (D^2)E\big)^2 \leq 0$

& $\big((DE)D - (D^2)E\big)^2 = 0 \Leftrightarrow (DE)D - D^2E \underset{num}{\equiv} 0$

taking $r = \frac{DE}{D^2}$.

$\square$