Exceptional Curves

A reduced, connected curve C on a smooth surface S is called exceptional if $\exists$ a birat'l morphism $\varphi : S \to S'$ such that C is φ-exceptional, that is, $\exists$ open nbhd U of C in X, $\exists$ a point $x \in S'$ & a nbhd V of x in S' s.t.

$$\begin{cases} \varphi : U \setminus C \xrightarrow{\cong} V \setminus \{x\} \\ \varphi(C) = x \end{cases}$$

In this case, we say that C is contracted to $x \in S'$.

Assume that $x \in S'$ is a normal point, this singularity is uniquely determined by the embedding $C \hookrightarrow S$ up to birat'l equiv.

We need the following characterization of exceptional curves, which is due to Grauert '62 & Mumford '61

THEOREM (Grauert's criterion)

a reduced, connected curve C with irred. components C_i on a smooth surface is exceptional $\Longleftrightarrow$ the intersection matrix $(C_i . C_j)$ is negative-definite.

Example (exceptional curves of the first kind)

a smooth rational curve C with $C^2 = -1$ is called a (-1)-curve. (or except'l curve of the first kind)

Prop (Characterization of (-1)-curves)

An irreducible curve $C \subset S$ is a (-1)-curve

$$\Longleftrightarrow \quad C^2 < 0 \ \& \ K_X C < 0.$$

Pf. • If C is a (-1)-curve, by adjunction formula

$$p_a(C) = \underset{\overset{\parallel}{0}}{g(C)} = 1 + \frac{1}{2}(K_X C + \underset{\overset{\parallel}{-1}}{C^2}) \Rightarrow K_X C = -1$$

• If $C^2 < 0$ & $K_X C < 0$, then $\deg(\omega_C) = C^2 + K_X C < 0$

$\Rightarrow p_a(C) = 1 + \frac{1}{2}\deg \omega_C \leq 0$ i.e. C smooth rational & $C^2 = -1$, $K_X C = -1$

$\square$

Prop | S smooth connected surface with Kodaira dim $\kappa(S) \geq 0$

$D \geq 0$ effective divisor on S s.t. $K_S . D < 0$

then D contains a (-1)-curve

Pf Suffices to show if D $\begin{cases} \text{irreducible curve} \\ \text{with } K_S D < 0 \end{cases}$, then D is a (-1)-curve

By assumption, $\kappa(S) \geq 0 \Rightarrow \exists$ a pluri-canonical divisor

$$nK_S = \sum_{i=1}^{t} r_i C_i \geq 0$$

$K_S D < 0 \Rightarrow D$ is one of the C_i's, say $D = C_1$

$\Rightarrow D(nK_S - r_1 C_1) \geq 0$ & $D^2 < 0$.

by the characterization of (-1)-curves, D is a (-1)-curve $\square$

② Hirzebruch-Jung strings.

let $C = \bigcup_{i=1}^{r} C_i$ with each C_i smooth rat'l curve. s.t.

$$\begin{cases} C_i^2 \leq -2, \ \forall i \\ C_i C_j = 1 \quad \text{if } |i-j| = 1 \\ C_i C_j = 0 \quad \text{if } |i-j| \geq 2. \end{cases}$$

denote $C_i^2 = e_i$.

then the configuration is visualized by the corresponding dual graph

$$e_1 - e_2 - e_3 \cdots e_{r-1} - e_r$$

the intersection matrix is

$$\begin{pmatrix} e_1 & 1 & 0 & \cdots & 0 \\ 1 & e_2 & 1 & \cdots & 0 \\ 0 & 1 & e_3 & \ddots & \\ & & \ddots & e_{r-1} & 1 \\ & & & 1 & e_r \end{pmatrix}$$

negative def.
$\downarrow$
< 0

The simplest H-J string is a smooth rat'l curve with C
$C^2 = -2$. (called (-2)-curve)

③ A-D-E Curves.

Def | An exceptional curve $C = \bigcup_{i=1}^{r} C_i$ on which all irreducible components C_i are (-2)-curves.

[Grauert's criterion]

Note $(C_i + C_j)^2 = 2(C_i C_j - 2) < 0$ for $\forall i \neq j$

$\Rightarrow C_i C_j \leq 1$

the intersection form of C is negative-definite

$\Rightarrow$ dual graphs of C are A-D-E type Dynkin diagrams.

Du Val singularities

Def. A point $p \in S$ of a normal surface S is said to be a Du Val singularity, if $\exists$ a smooth surface $\widetilde{S}$ & a morphism $\sigma : \widetilde{S} \longrightarrow S$ that contracts a reduced, connected curve $C = \bigcup_{i=1}^{r} C_i$ to a point p & $K_{\widetilde{S}} . C_i = 0$ for $1 \le i \le r$.

The surface $\widetilde{S}$ (as well as the morphism $\sigma : \widetilde{S} \to S$) is called the minimal desingularization/resolution of the singular point $p \in S$.

Prop Let $C \subset S$ be an exceptional curve & $K_S . C_i = 0$ for each irreducible component $C_i \subset C$, then C is an A-D-E curve.

Pf C exceptional curve $\overset{\text{Grauert's criterion}}{\Rightarrow}$ the intersection matrix $(C_i C_j)$ is negative definite.

$$\left.\begin{array}{l} K_S . C_i = 0 \\ C_i^2 < 0 \end{array}\right\} \overset{\text{by}}{\Rightarrow} \text{genus formula } p_a(C_i) = 1 + \tfrac{1}{2}(K_S . C_i + C_i^2)$$

$$p_a(C_i) = 0$$

$\Rightarrow$ each irreducible component C_i is smooth rational.

$\Rightarrow$ $C_i^2 = -2$ (i.e. $C = \bigcup_{i=1}^{r} C_i$ is a (-2)-curve.)

Consider the dual graph associated to the exceptional curve C

- Vertices $\longleftrightarrow$ irreducible components C_i
- two vertices are connected by a segment if the corresponding curves intersect.)

For $\forall i \ne j$, $(C_i + C_j)^2 = 2(C_i C_j - 2) < 0$

$\Rightarrow$ $C_i C_j \le 1$ for $\forall i \ne j$.

In fact, $p_a(C_i + C_j) = p_a(C_i) + p_a(C_j) + C_i C_j - 1$
$= C_i C_j - 1 \le 0 \quad (i \ne j)$
$\Rightarrow$ C is a tree of smooth rational curves

Fact (See Bourbaki'1968 Groupes et algèbres de Lie Chap 4-6 Chap VI - §4 Theorem 4)

Only the following graphs are possible:

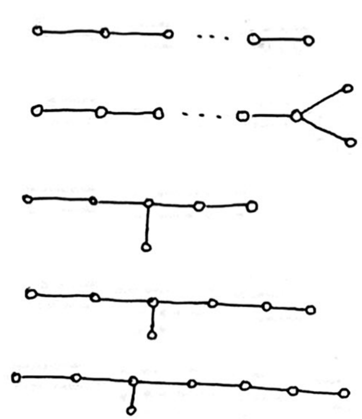
A_n ($n = \#$ vertices)

D_n

E_6

E_7

E_8



$\bigcup_{i=1}^{r} C_i$

Such a curve C called an A-D-E curve
$C_i^2 = -2$, $C_i K_S = 0$, $C_i C_j \le 1$ $(i \ne j)$

$\square$

The Du Val singularities can be given by the following equation

$$A_n : \quad x^2 + y^2 + z^{n+1} = 0$$

$$D_n : \quad x^2 + y^2 z + z^{n-1} = 0 \quad (n \geq 4)$$

$$E_6 : \quad x^2 + y^3 + z^{4} = 0$$

$$E_7 : \quad x^2 + y^3 + y z^3 = 0$$

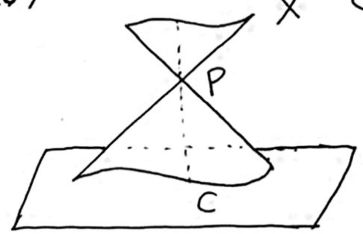
$$E_8 : \quad x^2 + y^3 + z^{5} = 0$$

<u>baby example</u> (the ordinary double point A_1)

(Ordinary) quadratic cone in 3-space (i.e. hypersurface (quadric)) defined by

$$\underset{(0,0,0)}{\overset{\parallel}{P}} \in X : (xz = y^2) \subset \mathbb{A}^3$$

$X \subset \mathbb{A}^3$



$\mathbb{P}^2$

X is an affine cone with vertex P and base the plane conic $C : (xz = y^2) \subset \mathbb{P}^2$

Consider the natural map

$$\theta : \mathbb{A}^3 - 0 \longrightarrow \mathbb{P}^2, \quad (x, y, z) \longmapsto [x : y : z]$$

$$\mathbb{P}^2 \supset C$$

$$X = \theta^{-1}(C) \cup \{P\}, \quad P = 0$$

The blow-up of X at P is the standard "cylinder" resolution $\varepsilon : \widetilde{X} \longrightarrow X$

$$\widetilde{X} = \left\{ (Q, \ell) \text{ pairs} \;\middle|\; \begin{array}{l} Q \in X \\ \ell \text{ a generating line through } P \end{array} \right\}$$

$$(t_1, t_2 ; x_1, x_2) \longmapsto \{(\lambda t_1, \lambda t_2 : \mu x_1, \lambda^{-2} \mu x_2) / (\lambda, \mu) \in (\mathbb{C}^*)^2\}$$

$$\begin{tikzcd}
(\mathbb{A}^2 \setminus 0) \times (\mathbb{A}^2 \setminus 0) \arrow[r] \arrow[d, "pr_1"'] & \mathbb{F}(0,2) = \mathbb{F}_2 = \widetilde{X} \arrow[d, "\pi"] \\
(\mathbb{A}^2 \setminus 0) \arrow[r] & \mathbb{P}^1
\end{tikzcd}$$

$(t_1, t_2) \qquad [t_1 : t_2]$

The exceptional curve $\varepsilon^{-1} P = E$ is a (-2)-curve, which is a negative section of $\pi : \mathbb{F}_2 \longrightarrow \mathbb{P}^1$

vertical part

F : fibre of $\pi \Rightarrow \operatorname{Pic} \mathbb{F}_2 = \mathbb{Z} E \oplus \mathbb{Z} F$

horizontal

$$\varphi : Y = \mathbb{F}_2 \xrightarrow{|2F+E|} \mathbb{P}^3 \qquad \begin{cases} F^2 = 0 \\ FE = 1 \\ E^2 = -2 \end{cases}$$

$$\operatorname{Im}\varphi = \overline{\mathbb{F}_2} = X \subset \mathbb{P}^3$$

φ is a contracting morphism contracting the negative section E

- A_1 as a quotient singularity

Consider the group action $G = \mathbb{Z}/2$ acting on $\mathbb{A}^2$ via
σ ($\sigma \in G$)

$$\sigma \curvearrowright \mathbb{A}^2 \longrightarrow \mathbb{A}^2$$
$$(u, v) \longmapsto (-u, -v)$$

$\Rightarrow$ quadratic monomials u^2, uv, v^2 are G-inv. fcts on $\mathbb{A}^2$

$$\Rightarrow \quad \mathbb{A}^2 \longrightarrow \mathbb{A}^3$$
$$(u, v) \longmapsto (\underset{x}{\underset{\shortparallel}{u^2}}, \underset{y}{\underset{\shortparallel}{uv}}, \underset{z}{\underset{\shortparallel}{v^2}})$$

identifies the quotient space $\mathbb{A}^2/G$ with $X_{(xz=y^2)} \subset \mathbb{A}^3$

Prop-Def
- G finite group acting on an affine variety V (by algebraic automorph.)
- $k[V]$ coordinate ring of V

then the quotient $X = V/G$ is an affine variety whose points correspond one-to-one with orbits of the group action. such that the polynomial functions on X are precisely the invariant poly. functions on V, i.e.

$$k[X] = k[V]^G$$

The quotient X ~~are~~ is just defined as follows:

$X = \operatorname{Spec}(k[V]^G)$, here $k[V]^G$ is the ring generated by finitely many G-invariant polynomials. Write

$$k[V]^G = \frac{k[u_1, u_2, \cdots, u_N]}{J} \quad \text{where} \begin{cases} u_i \text{ are the generators} \\ J \text{ ideal of relations between generators} \end{cases}$$

Then V/G is the subvariety of $\mathbb{A}^N_k$ (with coordinates $u_1, \cdots, u_N$) defined by the ideal J:

$$\underset{\mathbb{A}^N_k}{\overset{\cap}{X}} = V(J) = \{ u = (u_1, \cdots, u_n) \in k^N \mid f(u) = 0 \text{ for } \forall f \in J \}$$

Set-up : $V = \mathbb{C}^2$

$\underset{\text{generator}}{\langle \sigma \rangle} = G = \mathbb{Z}/r$ cyclic group of order r

$G \curvearrowright \mathbb{C}^2 \longrightarrow \mathbb{C}^2$

$\sigma . \quad (x, y) \longmapsto \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^a \end{pmatrix} \begin{pmatrix} x \\ y \end{pmatrix} = (\varepsilon x, \varepsilon^a y)$

Write $G = \left\langle \begin{pmatrix} \varepsilon & 0 \\ 0 & \varepsilon^a \end{pmatrix} \right\rangle$

here $\begin{cases} a \text{ integer coprime to } r \\ \varepsilon = \exp\left(\frac{2\pi i}{r}\right) \text{ primitive root of unity} \end{cases}$

- the singularities of $V/G = \mathbb{C}^2/(\mathbb{Z}/r)$ are called the surface cyclic quotient singularities of type $\frac{1}{r}(1, a)$

Rational singularities

Def | let $p \in X$ be a singular point & $f: Y \to X$ be a resolution of a normal surface singularity $p \in X$.
then P is called a rational singularity if $R^1 f_* \mathcal{O}_Y = 0$

Recall that given a continuous map $\varphi: X \to Y$, & a sheaf of abelian groups $\mathcal{F}$ on X, the i-th higher direct image of $\mathcal{F}$ defined to be the sheafification of the presheaf $(i \geq 0)$

$$\underset{\substack{\cap\ \text{open} \\ Y}}{V} \longmapsto H^i(\varphi^{-1}(V), \mathcal{F}|_{\varphi^{-1}(V)})$$

Consider the decomposition sequence for $(k+1)C = kC + C$

$$0 \to \mathcal{O}_C(-kC) \longrightarrow \mathcal{O}_{(k+1)C} \longrightarrow \mathcal{O}_{kC} \longrightarrow 0 \qquad (k \geq 1)$$

taking cohomology,

$$H^1(\mathcal{O}_C(-kC)) \longrightarrow H^1(\mathcal{O}_{(k+1)C}) \longrightarrow H^1(\mathcal{O}_{kC}) \to 0$$

<u>Grauert's result</u>

| $\exists$ arbitrary small strictly pseudo-convex nbhd U of C in X with the following property:
for $\forall$ locally free $\mathcal{O}_U$-sheaf $\mathcal{E}$, $\exists\, k_0 > 0$ s.t. for $\forall\, k \geq k_0$

$$\text{restr}: H^i(U, \mathcal{E}) \hookrightarrow H^i(\mathcal{E}|_{kC}) \quad (i \geq 1)$$

$$\text{restr}: H^1(U, \mathcal{E}) \cong H^1(\mathcal{E}|_{kC}) \quad (k \gg 0)$$

Hence

$$R^1 f_* \mathcal{O}_Y = 0 \iff h^1(\mathcal{O}_{kC}) = 0 \text{ for } \forall\, k \geq 1.$$

Prop | a (-1)-curve or a Hirzebruch-Jung string gives rise to a rational singularity.

Pf. by above decomposition sequence, suffices to show $h^1(\mathcal{O}_C(-kC)) = 0$ (for $k \geq 0$)

$$h^1(\mathcal{O}_C(-kC)) = h^0(\omega_C \otimes \mathcal{O}_C(kC))$$

- If C is a (-1)-curve, then C is smooth rational & $C^2 = -1$
  $\Rightarrow \mathcal{O}_C(C) = \mathcal{O}_{\mathbb{P}^1}(-1) \qquad h^1(\mathcal{O}_C(-kC)) = h^1(\mathcal{O}_{\mathbb{P}^1}(k)) = 0$
- If $C = \bigcup C_i$ is a Hirzebruch-Jung string, then

$$\deg(\underline{\omega_C \otimes \mathcal{O}_C(kC)}|_{C_i}) = \deg(\omega_{C_i}) + C_i(C - C_i) + kCC_i$$
$$= -2 - e_i + (k+1)CC_i$$

$$= \begin{cases} k-1+ke_i & \text{if } C_i \text{ is an end of the string} \\ 2k+ke_i & \text{Otherwise} \end{cases}$$

$\Rightarrow \deg(\omega_C \otimes \mathcal{O}_C(kC)|_{C_i}) \leq 0$

&

$\deg < 0$ if C_i an end curve

$\Rightarrow h^0(\omega_C \otimes \mathcal{O}_C(kC)|_{C_i}) = 0$. $\square$

THEOREM (Artin's criterion)

$C = \bigcup C_i$ an exceptional curve. TFAE

① C contracts to a rat'l singularity.

② each divisor $Z = \sum r_i C_i \geq 0$ has arithmetical genus $p_a(Z) := g(Z) \leq 0$.

③ $H^1(\mathcal{O}_Z) = 0$.

Pf "① ⇒ ③" for $\forall$ effective divisor Z, $\exists$ some $k > 0$ s.t.

$$\mathcal{O}_{kC} \twoheadrightarrow \mathcal{O}_Z.$$

$kC = Z + W$ $(W \geq 0)$ $\rightsquigarrow$ $0 \to \mathcal{O}_W(-Z) \to \mathcal{O}_{kC} \to \mathcal{O}_Z \to 0$

$\dim \mathrm{Supp}(C) = 1$ implies that

$$H^1(\mathcal{O}_Z) = 0 \Leftrightarrow H^1(\mathcal{O}_{kC}) = 0.$$

Since the singularity is rational, $H^1(\mathcal{O}_{kC}) = 0$

hence $H^1(\mathcal{O}_Z) = 0$.

"③ ⇒ ②" $p_a(Z) \overset{\text{by def.}}{=} 1 - \chi(\mathcal{O}_Z)$

$$= 1 - h^0(\mathcal{O}_Z) + h^1(\mathcal{O}_Z)$$

$$= 1 - h^0(\mathcal{O}_Z) \leq 0$$

"② ⇒ ①" Assume that for all effective divisor $Z = \sum r_i C_i$,

$$p_a(Z) = 0.$$

In particular, $p_a(C_i) = 0$ $\Rightarrow$ all C_i are smooth rational curves

Now use induction on $r = \sum r_i$ to show $h^1(\mathcal{O}_Z) = 0$.

So let $r \geq 2$ & $h^1(\mathcal{O}_{Z'}) = 0$ for all $Z' = \sum r_i' C_i$ with $\sum r_i' = r-1$.

let C_0 be a component of Z & $Z_0 = Z - C_0$,

Consider the decomposition sequence for $Z = Z_0 + C_0$,

$$0 \to \mathcal{O}_{C_0}(-Z_0) \to \mathcal{O}_Z \to \mathcal{O}_{Z_0} \to 0$$

taking cohomology,

$$\to H^1(\mathcal{O}_{C_0}(-Z_0)) \to H^1(\mathcal{O}_Z) \to H^1(\mathcal{O}_{Z_0}) \to 0$$

Suffices to show $h^1(\mathcal{O}_{C_0}(-Z_0)) = 0$

$\Leftrightarrow \quad Z_0 C_0 \leq 1$

$\Leftrightarrow \quad Z C_0 \leq 1 + C_0^2$

Since the C_0 is chosen arbitrarily, we have finished the proof unless $Z C_i \geq 2 + C_i^2$ for each C_i.

But in this case,

$$\begin{aligned} \deg(\omega_Z) &= \deg\left(K_X \otimes \mathcal{O}_X(Z)|_Z\right) \quad Z = \sum r_i C_i \\ &= \sum_i r_i \deg\left(K_X \otimes \mathcal{O}_X(Z)|_{C_i}\right) \\ &= \sum_i r_i (K_X C_i + Z C_i) \\ &= \sum_i r_i (Z C_i - C_i^2 - 2) \geq 0 \end{aligned}$$

$\Rightarrow p_a(Z) = 1 + \frac{1}{2}\deg(\omega_Z) > 0$ $\lightning$.

$\square$

by adjunction formula for embedded curves,

$$\omega_Z = \mathcal{O}_Z(K_X + Z)$$

Consider the residue sequence

$$0 \to K_X \to \underbrace{K_X \otimes \mathcal{O}_X(Z)}_{= K_X(Z)} \xrightarrow{\text{restr.}} \omega_Z \to 0$$

$h^0(\omega_Z) \overset{\text{Serre}}{=} h^1(\mathcal{O}_Z)$

we find that

Cor X surface, $Z \geq 0$ eff divisor supported on a curve which contracts onto a rat'l singularity

$\Rightarrow \quad H^0(K_X) \xrightarrow{\sim} H^0(K_X(Z))$.

Prop The A-D-E curves contract to rational singularities

pf. let $C = \bigcup C_i$ an exceptional curves with all C_i (-2)-curves

$\underset{= 0}{p_a(C_i)} = g(C_i) = 1 + \frac{1}{2}(\underset{= -2}{C_i^2} + K_X C_i) \Rightarrow K_X C_i = 0 \ \forall i.$

For $\forall$ effective divisor $Z = \sum r_i C_i$ (C exceptional)

$$\deg \omega_Z = \deg\left(K_X \otimes \mathcal{O}_X(Z)|_Z\right) = \sum_i r_i (K_X C_i + Z C_i) = Z^2 < 0$$

$\Rightarrow p_a(Z) = 1 + \frac{1}{2}\deg(\omega_Z) \leq 0 \Rightarrow Z$ contracted to a rat'l sing. $\square$

let $C = \bigcup C_i$ exceptional curve & U arbitrarily small nbhd of C in X with $H^1(U, \mathcal{O}_U^*) = H^2(U, \mathbb{Z})$ (say U affine)

& $$H^2(U, \mathbb{Z}) = H^2(C, \mathbb{Z}) = \bigoplus H^2(C_i, \mathbb{Z})$$

↑

free group with 1 generator for each C_i.

If C is an exceptional A-D-E curve, then $K_X C_i = 0$ $\forall i$.

Hence

<u>Prop</u> Any A-D-E curve has a neighborhood U with $K_U \cong \mathcal{O}_U$.

Consider the residual sequence

$$0 \to K_X \to K_X(Z) \to \omega_Z \to 0$$

twisted by $K_X^{\otimes(m-1)}$

$$0 \to K_X^{\otimes m} \to K_X^{\otimes m} \otimes \mathcal{O}_X(Z) \to (K_X^{\otimes m} + Z)|_Z \to 0$$

<u>Cor</u> X surface

$Z \geq 0$ a divisor supported on an A-D-E curve

$\Rightarrow$ for $\forall\, m \in \mathbb{Z}$,

$$H^0(K_X^{\otimes m}) \xrightarrow{\sim} H^0(K_X^{\otimes m}(Z))$$

Singularities of double coverings, simple surface singularities

<u>Set-up</u> : X normal surface (connected)

Y smooth, connected surface

$X \xrightarrow{\pi} Y$ double cover, ramified over $B \subset Y$.

Singularities of X lying over the singularities of B.

<u>Canonical resolution of X</u> :

let $y \in B$ singular point with multiplicity $\mu_y = \mathrm{mult}_y(B)$

$\sigma_1 : Y_1 \to Y$ blow-ups over all singularities $y \in B$. with exceptional curve $E_y := \sigma_1^{-1}(y)$ over y.

$$\Rightarrow \quad \sigma_1^* B = \hat{B} + \sum_{\substack{y \in B \\ \text{singular}}} \mu_y E_y \quad \text{total transform of } B$$

$\hat{B}$: strict transform of B

$$\begin{tikzcd} X_1 \arrow[r, "\nu"', "\text{normalization}"] \arrow[dr, "\pi_1"'] & X \times_Y Y_1 \arrow[r] \arrow[d] \arrow[dr, phantom, "\square"] & X \arrow[d, "\pi"] \\ & Y_1 \arrow[r, "\sigma_1"'] & Y \end{tikzcd}$$

$\pi_1 : X_1 \to Y_1$ is a double cover of Y_1, ramified over

$$B_1 = \hat{B} + \sum_{\mu_y \text{ odd}} E_y = \sigma_1^* B - \sum_y 2\left[\frac{\mu_y}{2}\right] E_y$$

If B_1 singular, repeat this construction & so on.

$\Rightarrow$ the new obtained ~~ramification~~ branch curves

$$B_1, B_2, \ldots, \subset \sigma^*(B)$$

$\sigma^*(B)$: total transform

THEOREM | X smooth surface, $C \subset X$ reduced curve

$\Rightarrow$ $\exists$ a smooth surface Y & a morphism $\varphi : Y \to X$ consisting of a finite number of blow-ups of X s.t.

① the reduction of total transform of C has only
- ordinary double points as singularities.

② proper transform of C is smooth.

by above THEOREM, after finitely many steps we arrive at a branch curve B_k with at worst nodes (i.e. ordinary double points)

$\Rightarrow$ all singularities y of B_k have multiplicities $\mu_y = 2$

$\Rightarrow$ $B_{k+1} = \overline{B_k} \leftarrow$ a smooth curve

$\Downarrow$

X_{k+1} is smooth & X_{k+1} is a resolution of the singularities of X.

$$\begin{tikzcd}
\text{Smooth} \arrow[r] & X_{k+1} \arrow[r, "\varphi_{k+1}"] \arrow[d, "\pi_{k+1}"'] & X_k \arrow[r] \arrow[d, "\pi_k"'] & \cdots \arrow[r, "\varphi_2"] & X_1 \arrow[r, "\varphi_1"] \arrow[d, "\pi_1"'] & X \arrow[d, "\pi"] \\
& Y_{k+1} \arrow[r, "\sigma_{k+1}"'] & Y_k \arrow[r] & \cdots \arrow[r, "\sigma_2"'] & Y_1 \arrow[r, "\sigma_1"'] & Y
\end{tikzcd}$$

$$B_i = \sigma_i^*(B_{i-1}) - 2\left[\frac{\mu_{y_i}}{2}\right] E_{y_i}$$

$\cap$ branch curve

Y_i

$\sigma_i : Y_i \longrightarrow Y_{i-1}$ blow-up of the singular point $y_{i-1} \in B_{i-1}$

$E_i := \sigma_i^{-1}(y_{i-1})$ exceptional curve

Def. The singularities of double covers branches over a curve B having an A-D-E singularity are called simple surface singularities.

by explicit equations for simple curve sing. we obtain the following explicit equations for simple surface sing:

A_n $(n \geq 1)$ $\quad w^2 + x^2 + y^{n+1} = 0$

D_n $(n \geq 4)$ $\quad w^2 + y(x^2 + y^{n-2}) = 0$

E_6 $\quad w^2 + x^3 + y^4 = 0$

E_7 $\quad w^2 + x(x^2 + y^3) = 0$

E_8 $\quad w^2 + x^3 + y^5 = 0$

$$\begin{tikzcd}
(w^2+f(x,y)=0) \subset \mathbb{A}^3 \arrow[r, "\pi"] \arrow[d, "pr"] & \mathbb{A}^2 \\
\mathbb{A}^2 &
\end{tikzcd}
\qquad
\begin{tikzcd}
(w,x,y) \arrow[d, mapsto] \\
(x,y)
\end{tikzcd}$$

double cover branched along B $\longrightarrow \pi$

$0 \in B \underset{\text{curve}}{\subset} \mathbb{A}^2$

singularity $(f(x,y)=0)$

THEOREM

Suppose ① $\pi : X \longrightarrow Y$ a double cover with X normal, Y smooth ramified over the reduced curve $B \subset Y$.

② L line bundle on Y satisfying $\mathcal{O}_Y(B) = L^{\otimes 2}$ (square root of $\mathcal{O}_Y(B)$) which determines the double cover $\pi : X = \mathrm{Spec}_{\mathcal{O}_Y}(\mathcal{O}_Y \oplus L^\vee) \longrightarrow Y$

Consider the canonical resolution diagram

$$\begin{tikzcd}
\text{Smooth} \arrow[r] & \tilde{X} \arrow[r, "\tau"] \arrow[d, "\tilde{\pi}"'] & X \arrow[d, "\pi"] \\
& \tilde{Y} \arrow[r, "\sigma"'] & Y
\end{tikzcd}$$

where $\begin{cases} \sigma : \tilde{Y} \longrightarrow Y \text{ a sequence of blow-ups over singularities of } B \\ \tilde{X} \text{ smooth} \end{cases}$

then $\exists$ effective divisor $Z \geq 0$ on $\tilde{X}$ with $\mathrm{Supp}(Z) \subseteq$ exceptional locus of τ

s.t. $K_{\tilde{X}} = (\pi\tau)^*(K_Y \otimes L) \otimes \mathcal{O}_{\tilde{X}}(-Z)$

Furthermore, $Z = 0 \Longleftrightarrow$ singularities of B are simple.

Pf. let $\sigma_1 : Y_1 \longrightarrow Y$ be the blow-up at a singular point $y \in B$ with multiplicity $\mu_y = \text{mult}_y(B)$ & exceptional curve E_y

then $\sigma_1^* B = \hat{B} + \mu_y E_y$, $K_{Y_1} = \sigma_1^* K_Y + E_y$

Consider
$$B_1 = \begin{cases} \hat{B} + E_y , & \text{if } \mu_y \text{ odd} \\ \hat{B} , & \text{Otherwise} \end{cases}$$
$$= \sigma_1^* B - 2 \left[\frac{\mu_y}{2} \right] E_y$$

if B_1 singular, repeat this process, say after t steps,

$$B_t = \sigma_t^* B_{t-1} - 2\left[\frac{\mu_{t-1}}{2}\right] E_t$$
$$= \underbrace{(\sigma_1 \cdots \sigma_t)^*}_{= \sigma} B - \sum_{i=1}^{t} 2\left[\frac{\mu_{i-1}}{2}\right] E_i = \sigma^* B - \sum_{i=1}^{t} 2\left[\frac{\mu_{i-1}}{2}\right] E_i$$

smooth

$$\underbrace{K_{Y_t} = \sigma_t^* K_{Y_{t-1}} + E_t = \sigma^* K_Y + \sum_{i=1}^{t} E_i}_{L_t} \qquad \mathcal{O}_Y(B) \cong L^{\otimes 2}$$

by abusing of notation

taking square root of $\mathcal{O}_Y(B_t)$

$$L_t \cong \sigma^* L \otimes \mathcal{O}_{Y_t}\left(-\sum_i \left[\frac{\mu_{t-1}}{2}\right] E_i\right)$$

In this case, the double cover $\pi_t : X_t \longrightarrow Y_t$ determined by (B_t, L_t) is smooth & $K_{X_t} = \pi_t^* (K_{Y_t} \otimes L_t)$

note $K_{Y_t} \otimes L_t = \sigma^*(K_Y \otimes L) \otimes \mathcal{O}_{Y_t}\left(\sum_i \left(1 - \left[\frac{\mu_{i-1}}{2}\right]\right) E_i\right)$

put $X_t = \tilde{X}$, $Y_t = \tilde{Y}$ $\quad \pi_t = \tilde{\pi}$

then
$$K_{\tilde{X}} = K_{X_t} = \tilde{\pi}^* (K_{Y_t} \otimes L_t)$$
$$= \tilde{\pi}^* \sigma^* (K_Y \otimes L) \otimes \tilde{\pi}^* \mathcal{O}_{\tilde{Y}}\left(\sum_i \left(1 - \left[\frac{\mu_{i-1}}{2}\right]\right) E_i\right)$$
$$= \tilde{\pi}^* \sigma^* (K_Y \otimes L) \otimes \mathcal{O}_{\tilde{X}}(-Z)$$

where $Z \geq 0$ & $\text{Supp}(Z) \subseteq$ exceptional locus for τ.

$Z = 0 \iff 1 - \left[\frac{\mu_y}{2}\right] = 0$ for all sing. y of all branch curves $B, B_1, \ldots, B_t$.

$\iff \mu_y = 2$ or 3 for all singularities

$\iff$ the singularities are only double or triple points.

By the classification of simple curve singularities. $0 \in (f(x,y)=0)$

double points $\longleftrightarrow$ $f(x,y) = x^2 + y^{n+1}$ $(n \geq 1)$ (type A_n)

triple points with 2 or 3 different tangents $\leftrightarrow$ $f(x,y) = y(x^2 + y^{n-2})$ $(n \geq 4)$ (type D_n)

triple points of simple triple points with 1 tangent

E_6 : $f(x,y) = x^3 + y^4 = 0$

E_7 : $f(x,y) = x(x^2 + y^3)$

E_8 : $f(x,y) = x^3 + y^5$

Summary The simple curves singularities are exactly the double points with equations $A_n : x^2 + y^{n+1} = 0$ $(n \geq 1)$

&

triple points with equations

$$D_n : \quad y(x^2 + y^{n-2}) = 0 \quad (n \geq 4)$$

$$E_6 : \quad x^3 + y^4 = 0$$

$$E_7 : \quad x(x^2 + y^3) = 0$$

$$E_8 : \quad x^3 + y^5 = 0.$$

$\square$

As an end of this section, we can prove that

A-D-E singularities = Simple Surface singularities

THEOREM

(1) A simple surface singularity is resolved by an except'l A-D-E curve of the corresponding type.

(2) An except'l A-D-E curve contracts to a simple surface sing. with the corresponding equation.

the singularity is determined uniquely by the corresponding dual graph.