Birational maps of surfaces

Blow-ups

set-up : | S nonsingular projective surface $/\mathbb{C}$
| $p \in S$

Then $\exists$ a smooth surface $\hat{S}$ & a morph. $\varepsilon : \hat{S} \to S$

Such that ① $\varepsilon|_{\varepsilon^{-1}(S-\{p\})} : \varepsilon^{-1}(S-\{p\}) \xrightarrow[\text{isom.}]{\cong} S$

② $\varepsilon^{-1}(p) = E \cong \mathbb{P}^1$

ε is unique up to isom.

Call ε is the blow-up of S at p

E : exceptional curve of the blow-up.

Construction of blow-ups

Take a neighborhood $U \ni p$ with local coordinates x, y at p (i.e. the curves $x=0$, $y=0$ intersect transversely at $p=(0,0)$)

Can shrink U if necessary we may assume p is the only point of U in the intersection $(x=0)\cap(y=0)$

define the subvariety $\underset{(x,y;[u:v])}{\hat{U}} \subset U \times \mathbb{P}^1_{[u:v]}$ by the equation

$$xv - yu = 0$$

$$\begin{tikzcd} \hat{U} \arrow[dr, "\varepsilon"'] & U \times \mathbb{P}^1 \arrow[d, "pr_2"] \\ p \in & \underset{(x,y)}{U} \subset \mathbb{A}^2 \end{tikzcd}$$

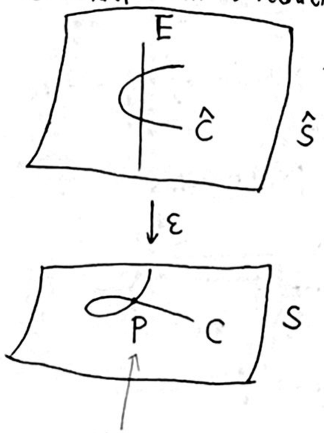
observe that

- $\varepsilon : \hat{U} \to U$ isom. over points of U where at most one of the coordinates x, y vanishes.
  $(x,y;[y:x]) \longleftarrow (x,y)$
- $\varepsilon^{-1}(p) \cong \mathbb{P}^1$
  $\| $
  $\{p\}\times\mathbb{P}^1$

$S = (S - \{p\}) \cup U \qquad (S-\{p\}) \cap U = U - \{p\}$

We get $\hat{S}$ by gluing $\hat{U}$ and $(S-\{p\})$ along $U - \{p\} \cong \hat{U} - \varepsilon^{-1}(p)$

Consider an irreducible curve C on S through p with multiplicity m. $\square$



ordinary double point

☆ Points of E identified with the tangent directions on S at p

$\underset{(x,y)}{\overset{\|}{p}} \in C = (f(x,y)=0)$

~~a=df~~

$p = (0,0) \qquad \prod L_i^{r_i} \leftarrow$ L_i distinct lines Called tangent lines to f at p

$f(x,y) = \underset{\text{form of deg } m}{f_m(x,y)} + f_{m+1} + \dots + f_n$

(if $\mathrm{mult}_p(C) = m$)

| if $r_i = 1$, L_i called simple tangent
| $r_i = 2$, double tangent.
| If f has m distinct simple tangents at p
| Say p ordinary multiple pt

$$\begin{tikzcd} E \arrow[d] & \subset \hat{S} \arrow[d, "\varepsilon"] \\ p & \in S \end{tikzcd}$$

C : irreducible curve on S through p

with multiplicity $\mathrm{mult}_p(C) = m$

$$\hat{C} := \overline{\varepsilon^{-1}(C - \{p\})} \subset \hat{S}$$

$\uparrow$

irreducible curve, called strict transform of C

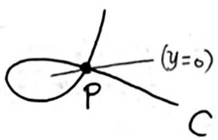
(proper transform)

lemma $\varepsilon^* C = \hat{C} + mE$

Pf clearly $\varepsilon^* C = \hat{C} + kE$ for some $k \in \mathbb{Z}_{\geq 0}$.

Choose local coordinates x, y in a neighborhood U of p

such that the curve $(y=0)$ is not tangent to any branch of C at p.



Then in $\widehat{\mathcal{O}_{S,p}}$, the equation of C can be written as a formal power series

$$f = f_m(x,y) + f_{m+1}(x,y) + \dots$$

where f_i : forms in x, y of degree i.

$m = \mathrm{mult}_p(C)$. $f_m(x,y) \neq 0$

$(xv - yu = 0)$

$$\begin{tikzcd} \hat{U} \arrow[r, hook] \arrow[dr, "\varepsilon"'] & U \times \mathbb{P}^1 \arrow[d, "pr_1"] \subset \mathbb{A}^2_{(x,y)} \times \mathbb{P}^1_{[u:v]} \\ & U_{(x,y)} \subset \mathbb{A}^2 \end{tikzcd}$$

$p \in U$, $p = (0,0)$

in a neighborhood of the point $(p, \infty) \in \hat{U} \subset \hat{S}$, where $(p,\infty) = (p, [0:1])$

can take the functions x and $t = v/u$ as local coordinates.

then $\quad xv = yu \Leftrightarrow x\frac{v}{u} = y \Leftrightarrow \boxed{xt = y}$

$$\varepsilon^* f(x,y) = f(x, tx) \qquad f = f_m + f_{m+1} + \dots$$

$$= x^m \underbrace{[f_m(1,t) + x f_{m+1}(1,t) + \dots]}$$

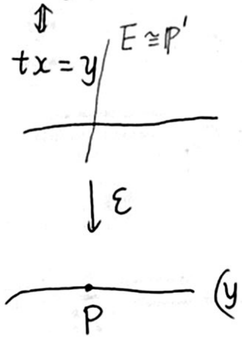
$\uparrow$

defining $\hat{C}$

$xv = yu$

$\Updownarrow$

$tx = y$



$(y=0) \Rightarrow tx = 0$

$t \in \mathbb{P}^1$

$(p,\infty) \in \hat{U}$ $(x\overset{1}{v} - y\overset{0}{u} = 0)$

exceptional curve $E = \varepsilon^{-1}(p) \subset \hat{U}$

$\parallel$

$(x = 0)$

$\Rightarrow k = m$

$$\Rightarrow \varepsilon^* C = \hat{C} + mE$$

$\square$

Prop. S surface

$\varepsilon : \hat{S} \longrightarrow S$ the blow-up of a point $p \in S$

$E \subset \hat{S}$ exceptional curve. Then

(1) $\exists$ an isomorphism $\mathrm{Pic}\, S \oplus \mathbb{Z} \xrightarrow{\sim} \mathrm{Pic}\, \hat{S}$

$$(D, n) \longmapsto \varepsilon^* D + nE$$

(2) D, D' divisors on S, then

$$\left|\begin{array}{l} \varepsilon^* D . \varepsilon^* D' = D . D' \\ E . \varepsilon^* D = 0 \\ E^2 = -1 . \end{array}\right.$$

(3) $NS(\hat{S}) \cong NS(S) \oplus \mathbb{Z} \cdot [E]$

(4) $K_{\hat{S}} = \varepsilon^* K_S + E$.

Pf. (2) Recall that the intersection pairing is defined on Picard group, can replace D and D' by linearly equivalent divisors.

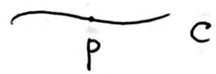
So we may assume that p doesn't lie on components of D & D'. Hence

$$\left|\begin{array}{ll} \varepsilon^* D . \varepsilon^* D' = D . D' & \leftarrow \varepsilon \text{ isom. outside } p \\ E . \varepsilon^* D = 0 & \leftarrow D \text{ not passes through } p \end{array}\right.$$

Choose a curve C passing through p with multiplicity 1.

$\Rightarrow$ the strict transform $\hat{C}$ meets E transversely at one point.

Corresponding to the tangent direction of C at p.



$$\left.\begin{array}{r} \Rightarrow \hat{C} . E = 1 \\ \varepsilon^* C = \hat{C} + E \end{array}\right\} \Rightarrow \underset{0}{\underset{\|}{\varepsilon^* C . E}} = \underset{1}{\underset{\|}{\hat{C} . E}} + E^2$$

$$\Downarrow$$

$$E^2 = -1$$

(1). $\forall$ irreducible curve on $\hat{S}$ (other than E) is the strict transform of its image in S.

$\Rightarrow$ the map $\mathrm{Pic}\, S \oplus \mathbb{Z} \longrightarrow \mathrm{Pic}\, \hat{S}$

$$(D, n) \longmapsto \varepsilon^* D + nE$$

is surjective.

· Suppose that $\exists$ divisor $D \subset S$ such that $\underset{\mathrm{Pic}\, \hat{S}}{\underset{\Uparrow}{\varepsilon^* D + nE = 0}}$

$\Rightarrow \underset{-n}{\underset{\|}{(\varepsilon^* D + nE) E}} = 0 \quad \Rightarrow n = 0$

$\Rightarrow 0 = \varepsilon_* \varepsilon^* D = D \in \mathrm{Pic}\, S$

(3) note that ε_* & ε^* are defined on the Néron-Severi groups

&
$$\begin{tikzcd} \mathrm{Pic}\,S \times \mathrm{Pic}\,S \arrow[r, "\cdot"] \arrow[d, "c_1 \times c_1"'] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{Z} \arrow[d, "\cong"] \\ NS(S) \times NS(S) \arrow[r, "\cup"', "\text{cup-product}"] & H^4(S, \mathbb{Z}) \end{tikzcd}$$

(4) Choose a meromorphic 2-form ω on S such that ω is holomorphic in a neighborhood of p & $\omega(p) \neq 0$.

It's clear that away from E the zeros and poles of $\varepsilon^*\omega$ are those of ω (via ε^*).

$\Rightarrow$ $\mathrm{div}(\varepsilon^*\omega) = \varepsilon^* \mathrm{div}(\omega) + kE$ for some $k \in \mathbb{Z}$.

i.e. $\varepsilon^* K_S + kE = K_{\hat{S}}$. By genus formula

$$\underbrace{g(E)}_{\overset{\|}{0}} = 1 + \frac{1}{2}(\underbrace{E^2}_{\overset{\|}{-1}} + \underbrace{K_{\hat{S}}.E}_{\overset{\|}{-k}}) \Rightarrow k = 1.$$

Alternatively, if $\omega = dx \wedge dy$ where x, y local coordinates at $p \in S$

$y = tx$ then $\varepsilon^*\omega = dx \wedge d(tx) = x\, dx \wedge dt$ in local coordinates x, t at a point of $E \subset \hat{S}$

$\Rightarrow \varepsilon^* K_S + E = K_{\hat{S}}$.

Rational maps ~~& linear systems~~

rat'l maps

Set-up X, Y varieties with X irreducible

a rational map $\phi: X \dashrightarrow Y$ is a morphism $\begin{matrix} U \longrightarrow Y \\ \cap \text{ open} \\ X \end{matrix}$

which cannot be extended to any larger open subset.

We say that ϕ is defined at x if $x \in U$.

Suppose that S is a smooth surface & $\varphi: S \dashrightarrow Y$ rat'l map

then the undefined set of φ, $\Sigma := S - U$, is a finite set.
(Called indeterminancy locus of φ)

Prop | X normal variety (e.g. smooth), Y projective variety
$\varphi: X \dashrightarrow Y$ a rational map
then the indeterminacy locus of φ has codim ≥ 2.

Pf $X \dashrightarrow Y \hookrightarrow \mathbb{P}^n$

We can reduce to the case $Y = \mathbb{P}^n$

Now consider rational map $\varphi: X \dashrightarrow \mathbb{P}^n$

The question is local.

for $\forall$ point x in the indeterminacy locus of φ.

X normal $\Rightarrow$ $\mathcal{O}_{x,X}$ integrally closed domain

For any codim 1 component Z of indeterminacy locus of φ passing through x, then $\mathcal{O}_{X,Z}$ is a DVR, say Z is defined by a single equation $g \in \mathcal{O}_{X,x}$.

$\varphi: X \dashrightarrow \mathbb{P}^n$ given by $(\varphi_0, \varphi_1, \cdots, \varphi_n)$ with $\varphi_i \in k(X)$ rational functions

Can multiply by a common factor (in $k(X)$) such that these φ_i no common factor & $\varphi_i \in \mathcal{O}_{X,x}$

$\Rightarrow$ the indeterminacy locus of φ in a neighborhood of x is the common zero locus

$$\bigcap_{i=0}^{n} (\varphi_i = 0).$$

$\Rightarrow$ g is a common factor of these φ_i, contradicting to the choice of the φ_i. $\square$

In particular,

$\varphi : C \dashrightarrow \mathbb{P}^n$ rational map, C smooth curve $\Rightarrow$ φ is a morphism

$\varphi : S \dashrightarrow \mathbb{P}^n$ rational map, S smooth surface $\Rightarrow$ indeterminacy locus of φ is a finite set of points of S

Now let $\varphi : S \dashrightarrow Y$ be a rational map, where

S a smooth surface & Y projective variety, &
Σ the indeterminacy locus of φ

If $C \subset S$ an irreducible curve, then φ defined on $C - \Sigma$

In this case, the image of C under φ defined to be

$$\varphi(C) := \overline{\varphi(C - \Sigma)} \subset Y$$

taking closure

Similarly, $\varphi(S) := \overline{\varphi(S - \Sigma)} \subset Y$

Note that $\forall$ codim 2 subset does not affect the Picard groups

that is, $\operatorname{Pic} S \xrightarrow[\text{restr.}]{\cong} \operatorname{Pic}(S - \Sigma)$

$$\left.\begin{array}{l} \operatorname{Pic} S \xrightarrow[\text{restr.}]{\cong} \operatorname{Pic}(S - \Sigma) \\ \varphi : S - \Sigma \longrightarrow Y \rightsquigarrow \operatorname{Pic} Y \xrightarrow{\varphi^*} \operatorname{Pic}(S - \Sigma) \end{array}\right\} \Rightarrow$$

$$\begin{tikzcd} \operatorname{Pic} Y \arrow[r] \arrow[rr, bend right, "\text{still denote } \varphi^*"'] & \operatorname{Pic}(S - \Sigma) \arrow[r, "\cong"] & \operatorname{Pic} S \end{tikzcd}$$

linear systems.

set-up

S : surface

D: divisor on S, say $D = \sum n_i C_i$

let

$$|D| := \left\{ D' \geq 0 \;\middle|\; \begin{array}{l} D' \text{ effective divisor on } S \\ \text{with } D \underset{lin}{\sim} D' \end{array} \right\}$$

Called the linear system associated to D. By definition,

for $\forall\, D' \in |D|$, $\exists$ a rational function $0 \neq f \in k(S)$ s.t.

$$D' = D + \operatorname{div}(f)$$

Such a section $f \in k(S)$ determined uniquely up to a scalar

$\Rightarrow$ if we consider

$$L(D) := \{ f \in k(S)^* \mid \operatorname{div}(f) + D \geq 0 \} \cup \{0\}$$

then can identify

$$|D| \cong \mathbb{P}(L(D))$$

<u>Rmk</u> $L(D)$ is a vector space which is the set of all rational sections/functions of S having order $\geq -n_i$ along C_i.

For $\exists_a$ $f_0 \in H^0(S, \mathcal{O}_S(D))$ with $\operatorname{div}(f_0) = D$, then

for $\forall\, f \in H^0(S, \mathcal{O}_S(D))$, the quotient $t_f := f/f_0 \in k(S)$

with $\operatorname{div}(t_f) = \operatorname{div}(f) - \operatorname{div}(f_0) \geq -D$, i.e. $t_f \in L(D)$

& $\operatorname{div}(f) = D + \operatorname{div}(t_f) \geq 0$

$$\begin{array}{c} \cap \\ |D| \end{array}$$

Conversely, for $\forall\, t_f \in L(D)$

$$s := t_f \cdot f_0 \in H^0(S, \mathcal{O}_S(D))$$

$\Rightarrow$ We have an identification

$$L(D) \xrightarrow{\otimes f_0} H^0(S, \mathcal{O}_S(D))$$

<u>Summary</u>

$$|D| \cong \mathbb{P}(L(D)) \cong \left. H^0(S, \mathcal{O}_S(D)) - \{0\} \right/ \mathbb{C}^*,$$

a linear subspace $P \subset |D|$ called a linear (sub-)system

$\updownarrow$ corresp.

a subvector space $V_P \subset H^0(\mathcal{O}_S(D))$

We say the linear system P is complete if $P = |D|$

$\dim |P| := \dim_{\mathbb{C}} \mathbb{P}(V_P)$

linear systems of dim 1, 2, or 3 called pencils, nets or webs, respectively.

let P be a linear system on S, a curve C is called a <u>fixed component</u> of P if for $\forall$ divisor $D \in P$, $C \subseteq D$.

The <u>fixed part</u> of P is the biggest divisor Z with $Z \subseteq D$ for $\forall D \in P$

A point $x \in S$ called a <u>base point</u> of P if for $\forall D \in P$ $x \in D$

Collecting all base points of P, define the <u>base locus</u> of P as

$$Bs(P) := \{x \in S \mid x \in \operatorname{Supp} D, \text{ for } \forall D \in P\}$$

For surface S and linear system P on S, let Z be the fixed part of P (if any), then $P - Z$ is a linear system M having no fixed part & only a finite number of base points.

i.e. $P = M + Z$ ← moving/mobile part of P

Clearly, in this case

$$\#\{\text{base points of } M\} \leq D^2 \text{ for } D \in \overset{\text{no fixed part.}}{M}$$

<u>Bertini's Theorem</u>

Let P be a linear system on a smooth variety X & $D \in P$ a general member, then D is smooth outside $Bs(P)$.

<u>Pf</u> If the generic element of P is singular away from $Bs(P)$ then the generic element of a generic pencil $\subseteq P$ will be singular away from $Bs(P)$

$\Rightarrow$ suffices to prove Bertini for a pencil $\{D_\lambda\}$

The question is local. We may assume locally general member

$$D := D_\lambda = \{f(x_1, \cancel{x_2}, \dots, x_n) + \lambda g(x_1, x_2, \dots, x_n) = 0)$$

$0 \in \operatorname{Supp} D_{\cancel{\lambda}}$ is a singular point & $0 \notin Bs|D_{\cancel{\lambda}}|$

then $f(0) \neq 0$ (if $f(0)=0$, $0 \in D \Rightarrow g(0)=0 \Rightarrow 0 \in Bs|D_\lambda|$)

$\Rightarrow g(0) \neq 0 \Rightarrow \lambda = -\frac{f(0)}{g(0)}$

D singular at $0 \Leftrightarrow \left(\frac{\partial f}{\partial x_i} - \lambda \frac{\partial g}{\partial x_i}\right)\Big|_{x=0} = 0$ for $\forall i \Rightarrow \frac{\partial}{\partial x_i}\left(\frac{f}{g}\right)\Big|_{x=0} = 0 \quad \forall i$

$\Rightarrow \frac{f}{g}$ constant on $\forall$ <u>connected</u> (finitely many) component of singular locus D_{sing}, outside $Bs|D| \Rightarrow \exists$ finitely many divisors meeting D

Rational maps & linear systems

S : surface

Then $\exists$ a bijection

$$\left\{\begin{array}{l}\text{rat'l maps } \varphi : S \dashrightarrow \mathbb{P}^n \text{ s.t.} \\ \phi(S) \text{ non-degenerate}\end{array}\right\} \overset{1-1}{\longleftrightarrow} \left\{\begin{array}{l}\text{linear systems } P \text{ on } S \\ \text{without fixed part } \& \dim P = n\end{array}\right\}$$

In fact, given a rational map $\varphi : S \dashrightarrow \mathbb{P}^n$, let $|\mathcal{O}_{\mathbb{P}^n}(1)|$ be the linear system of hyperplanes in $\mathbb{P}^n$, then $\varphi^*|\mathcal{O}_{\mathbb{P}^n}(1)|$ is a linear system of dim n & no fixed part (since it comes from the $|\mathcal{O}_{\mathbb{P}^n}(1)|$, which has no fixed part)

Conversely, given a linear system P on S without fixed part & of dim n. let $\check{P}$ be the projective space dual to P.

Now define a rational map

$$\begin{aligned} \phi : S &\dashrightarrow \check{P} \\ x &\longmapsto \text{hyperplane } H_x \in P \end{aligned}$$

consisting of the divisors through x

$\Rightarrow$ ϕ defined at x $\Leftrightarrow$ x not a base point of P

Thm (elimination of indeterminacy)

let $\varphi : S \dashrightarrow X$ be a rational map, where

S : surface, X projective variety

then $\exists$ a surface S' & a morphism $\eta : S' \to S$ (which is the composition of a finite number of blow-ups) & a morphism $f : S' \longrightarrow X$ such that the following diagram commutes

$$\begin{tikzcd} S' \arrow[d, "\eta"', "\text{blow-ups}" near start] \arrow[dr, "f\ (\text{morphism})"] \arrow[dr, phantom, "\circlearrowleft" near start, shift right=3ex] & \\ S \arrow[r, dashed, "\varphi"] & X \end{tikzcd}$$

Pf X projective $\Rightarrow X \hookrightarrow \mathbb{P}^m \Rightarrow$ WMA : $X = \mathbb{P}^m$

We may also assume $\varphi(S)$ nondegenerate, otherwise, consider X as a lower-dim'l projective space. $\varphi : \cancel{X}S \dashrightarrow \mathbb{P}^m$ rat'l map

$\Rightarrow$ φ corresponds to a linear system $P \subset |D|$ of dim m on S without fixed part.

① If P has no fixed point, then φ is a morphism (everywhere defined)
We are done!

② Suppose that P has a base point, say $x \in S$.

then consider the blow-up at x (with $E \subset S_1$ the exceptional curve of ε)

$$\begin{array}{c} \varepsilon_1 : S_1 \longrightarrow S \\ \parallel \\ Bl_x S \\ E \longmapsto x \end{array}$$

$x \in P$ base point $\Rightarrow$ $E \subset \left(\text{fixed part of } \varepsilon_1^* P \subset |\varepsilon_1^* D|\right)$ with multiplicity $k \geq 1$

$\Downarrow$

let $P_1 := \varepsilon_1^* P - kE \subset |\varepsilon_1^* D - kE|$ be the sub-system obtained by subtracting kE from each member of $\varepsilon_1^* P$

& P_1 no fixed component (since P no fixed part & $\varepsilon_1 = Bl_x S$)

$\Downarrow$

P_1 defines a rational map $\phi_1 : S_1 \dashrightarrow \mathbb{P}^m$ & $\phi_1 = \phi \circ \varepsilon_1$

If P_1 no base points, we are done.

Otherwise, repeat the process.

$\Downarrow$

We get a sequence of blow-ups $\varepsilon_n : S_n \longrightarrow S_{n-1}$ & linear systems P_n on S_n satisfying $P_n \subset |D_n|$ & P_n no fixed part. &

$$D_n = \varepsilon_n^* D_{n-1} - k_n E_n \quad \& \quad \varphi_n : S_n \dashrightarrow \mathbb{P}^m$$

Now consider the intersection numbers

- $D_n^2 = (\varepsilon_n^* D_{n-1} - k_n E_n)^2 = D_n^2 - k_n^2 < D_{n-1}^2$
- P_n no fixed part $\Rightarrow$ $D_n^2 \geq 0$ for all n
- $0 \leq D_n^2 < D_{n-1}^2 < \cdots < D^2$

finite number

$\Longrightarrow$ the process above must terminate

i.e. eventually we obtain a system P_n without base points which defines a morphism $f := \varphi_n : S \longrightarrow \mathbb{P}^n$, as required

<u>Rmk</u>

By the proof, we know how to construct S' & the morphism f $\square$

& $\#\{\text{blow-ups required}\} \leq D^2$

Structure of birational maps

Prop (Universal property of blow-ups)

$f: S \to T$ birational morphism of surfaces

Suppose the rat'l map $f^{-1}: T \dashrightarrow S$ is undefined at $t \in T$

then f factorizes as

$$\begin{tikzcd} & \hat{S} \arrow[dr, "\varepsilon"] & \\ S \arrow[ur, "g"] \arrow[rr, "f"'] & & T \end{tikzcd}$$

where $g: S \to \hat{S}$ birational morphism

$\varepsilon: \hat{S} \to T$ blow-up at $t \in T$

Observation 1

S irreducible surface

S' smooth surface

$f: S \to S'$ birational morphism

Suppose the rational map $f^{-1}: S' \dashrightarrow S$ is undefined at $p \in S'$

then $f^{-1}(p)$ is a curve on S

Pf The question is local on S, we may assume S affine with $f^{-1}(p) \neq \phi$, then $S \overset{j}{\hookrightarrow} \mathbb{A}^n$ embedding

$$\begin{tikzcd} S' \arrow[r, dashed, "f^{-1}"'] \arrow[rr, bend left, "j\circ f^{-1}"] & S \arrow[r, hook, "j"'] & \mathbb{A}^n \end{tikzcd}$$

coordinate $(x_1, \dots, x_n)$ with $x_i \in \mathbb{C}$.

the rat'l map $j \circ f^{-1}: S' \dashrightarrow \mathbb{A}^n$ defined by $(g_1, \dots, g_n)$

where $g_i \in K(S')$ rational functions.

f^{-1} not defined at $p \in S' \Rightarrow$ one of g_i not defined at $p \in S'$

Say g_1 not defined at $p \in S'$.

i.e. $g_1 \notin \mathcal{O}_{S',p}$ ~~not~~ rat'l fcts defined at p

S' smooth $\Rightarrow \mathcal{O}_{S',p}$ regular local ring

$\Downarrow$

UFD

Write $g_1 = u/v$ with $\begin{cases} u, v \in \mathcal{O}_{S',p} \\ u, v \text{ coprime} \\ v(p) = 0 \end{cases}$

Consider the curve $D: (f^*v = 0) \subset S$

then $x_1 = f^*(g_1) = \dfrac{f^*u}{f^*v}$ i.e. $f^*u = x_1 f^*v$ on S

first cordinate function on $S \subset \mathbb{A}^n$

$\Rightarrow$ On $D: (f^*v = 0)$, one has $f^*v = f^*u = 0$

$\Rightarrow D = f^{-1}((u = v = 0))$ $\quad u, v$ coprime $\Rightarrow (u=0) \cap (v=0)$ finite set

Shrinking S' if necessary, we can assume $(u = v = 0)$ is a single point $\{p\}$ $\square$

Observation 2

$\varphi : S \dashrightarrow S'$ rational map of surfaces s.t.

$\varphi^{-1} : S' \dashrightarrow S$ undefined at $p \in S'$

$\Rightarrow$ $\exists$ a curve $C \subset S$ such that $\varphi(C) = \{p\}$

Pf φ corresponds to a morphism $f : U \longrightarrow S'$, $U \underset{\text{open}}{\subset} S$

let $\Gamma_f \subset U \times S'$ be the graph of f

$\Gamma_f = \{(x, f(x)) \in U \times S' \mid x \in U\}$

& $S_1 := \overline{\Gamma_f} \subset S \times S'$ (closure)

then S_1 an irreducible surface.

$S_1 \subset S \times S'$

$$\begin{tikzcd} & S_1 \arrow[dl, "p"'] \arrow[dr, "q"] \arrow[d, phantom, "\circlearrowright"] & \\ S \arrow[rr, dashed, "\varphi"'] & & S' \end{tikzcd}$$

projections

two projections p, q are birational morphisms.

Since $\varphi^{-1} : S' \dashrightarrow S$ undefined at $p \in S'$

$\Downarrow$

$$\begin{tikzcd} & S_1 \arrow[dl, "p"'] \arrow[dr, "q"] & \\ S \arrow[rr, dashed, "\varphi"] & & S' \arrow[ul, dashed, bend right, "q^{-1}"'] \arrow[ll, bend left, "\varphi^{-1}"] \end{tikzcd}$$

p

φ^{-1} rational map

q^{-1} undefined at $p \in S'$

$\Downarrow$ Observation 1

$\exists$ an irred. curve $C_1 \subset S_1$ s.t. $q(C_1) = \{p\}$

$\Downarrow$

$S_1 \subset S \times S'$

$$\begin{tikzcd} & S_1 \arrow[dl, "p"'] \arrow[dr, "q"] & \\ S & & S' \end{tikzcd}$$

$C := p(C_1) \subset S$ curve

& $\varphi(C) = \{p\}$ $\square$

proof of Prop

$$\left(\begin{tikzcd} & \hat{S} \arrow[dr, "\varepsilon\ \text{blow-up at } t"] \arrow[d, phantom, "\circlearrowright"] & \\ S \arrow[ur, "\text{bir}", "g"'] \arrow[rr, "f"] & & T \arrow[ll, dashed, bend left, "f^{-1}", "\text{bir}"'] \end{tikzcd} \right)$$

$t \in T$ undefined

Consider the birational maps $g := \varepsilon^{-1} \circ f : S \dashrightarrow \hat{S}$ & $s = g^{-1} = f^{-1} \circ \varepsilon : \hat{S} \dashrightarrow S$

Suppose that g undefined at $x \in S$ $\overset{\text{obser. 2}}{\Longrightarrow}$ $\exists$ curve $C \subset \hat{S}$ s.t. $s(C) = x$, $s(C) = f^{-1}(\varepsilon(C))$

$\Downarrow$

$\varepsilon(C) = f(x)$ point, $\varepsilon(C) = E$, $f(x) = t$

Claim: $\boxed{\exists \text{ a local coordinate } v \text{ on } T \text{ at } t \text{ such that } f^*v \in \mathfrak{m}_x^2}$

Indeed, let (z, w) be a local coordinate system at $\cancel{\in S'}$ $t \in T$

if $f^*w \notin \mathfrak{m}_x^2 \Rightarrow f^*w$ vanishes on $f^{-1}(t)$ with multiplicity 1

$\Rightarrow$ defines a local equation for $f^{-1}(t)$ in $\mathcal{O}_{S,x}$

$\Rightarrow$ $f^*z = u f^*w$ for some $u \in \mathcal{O}_{S,x}$

Put $y = z - u(x)\cdot w$

$$\Rightarrow f^*y = f^*z - u(x) f^*w = (u-u(x)) f^*w \in \mathfrak{m}_x^2 .$$

end of claim $\square$

Then let $e \in E$ be a point where the map $s : \hat{S} \dashrightarrow S$ is defined.

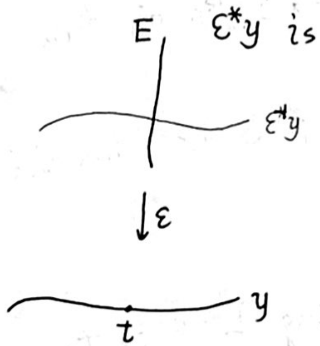
$$\begin{array}{ccc} \hat{S} & \dashrightarrow & S \\ \cup & & \\ E & & \\ \in & & \\ e & \longmapsto & s(e) \end{array}$$

We have

$$s^* f^* y = \varepsilon^* y \in \mathfrak{m}_e^2$$

holds for all $e \in E$ outside the finite set (i.e. indeterminacy locus of s)

By construction of blow-up,



$\varepsilon^* y$ is a local coordinate at $\forall$ point of E except one

$\lightning$ $\square$

Thm (Structure of birational morphisms)

$f : S \to S_0$ birat'l morphism of surfaces

$\Rightarrow$ $\exists$ a sequence of blow-ups $\varepsilon_k : S_k \to S_{k-1}$ $(1 \le k \le n)$

& $\exists$ an isomorphism $\theta : S \xrightarrow{\sim} S_n$ such that f factorizes as

$$\begin{tikzcd} & S_n \arrow[r, "\varepsilon_n"] & \cdots \arrow[r] & S_2 \arrow[r, "\varepsilon_2"] & S_1 \arrow[dr, "\varepsilon_1"] & \\ S \arrow[ur, "\theta", "\sim"'] \arrow[rrrrr, "f"'] & & & \circlearrowright & & S_0 \end{tikzcd}$$

Pf. If f an isom, we are done.

Otherwise, $\exists$ a point $p \in S_0$ s.t. f^{-1} undefined at p.

By universal property of blow-up, f factorizes as

birat'l morph. $\to$ f_1; ε_1 $\leftarrow$ blow-up at p

$$\begin{tikzcd} & S_1 \arrow[d, "\varepsilon_1"] \\ S \arrow[ur, "f_1"] \arrow[r, "f"'] & S_0 \end{tikzcd}$$

If the result fails, then we can construct an infinite sequence of blow-ups $\varepsilon_k : S_k \to S_{k-1}$ & birat'l morphisms $f_k : S \to S_k$ s.t. $\varepsilon_k \circ f_k = f_{k-1}$.

denote

$$n(f_k) := \# \left\{ \begin{array}{l} \text{irreducible curves on } S \\ \text{contracted by } f_k \end{array} \right\}$$

Clearly

$$\begin{tikzcd} & S_k \arrow[r, "\varepsilon_k"] & S_{k-1} \arrow[r] & \cdots \arrow[d] \\ S \arrow[ur, "f_k"] \arrow[urr, "f_{k-1}"'] \arrow[rrr] & & & S_0 \end{tikzcd}$$

- $\forall$ f_k-contracted curve is also contracted by f_{k-1}.
- $\exists \geq 1$ irreducible curve $C \subset S_k$ s.t. $f_k(C)$ $\underset{\text{exceptional curve of } \varepsilon_k}{\parallel}$

$\Longrightarrow$ Such a curve $C \subset S$ is contracted by f_{k-1}, but not by f_k

$\Longrightarrow$ $n(f_k) < n(f_{k-1})$

When k tends to infinity, $n(f_k) < 0$ $\lightning$. $\square$

Immediately, we also obtain the structure of birat'l maps of surfaces.

Cor. let $\varphi: S \dashrightarrow S'$ be a birational map of surfaces

then $\exists$ a surface $\hat{S}$ & a commutative diagram

$$\begin{tikzcd} & \hat{S} \arrow[dl, "f"'] \arrow[dr, "g"] & \\ S \arrow[rr, dashed, "\varphi"'] & & S' \end{tikzcd}$$

Where f, g are compositions of blow-ups & isom. $\square$

(by the universal property of blow-ups & structure of birat'l morphisms)

Remarks

(1) If $f: S \longrightarrow S'$ birat'l morph. of surfaces.

Say f is a composition of n blow-ups & an isom.

then $NS(S) \cong NS(S') \oplus \mathbb{Z}^n$ $\Rightarrow \rho(S) = \rho(S') + n$

$NS(S) \subset H^2(S, \mathbb{Z})$, finitely generated; $\rho(S) \leq b_2(S)$, finite number

$\Longrightarrow$ n is uniquely determined & independent of the chosen factorization

$\Longrightarrow$ $\forall$ birat'l morph (of S) $S \longrightarrow S$ is an isom.

(2) The blow-up $\varepsilon: \hat{S} \longrightarrow S$ at a point $p \in S$ also has the following universal property:

if $f: \hat{S} \longrightarrow X$ (variety) a morph. contracting $E = \varepsilon^{-1}(p)$ to a point

then f factors through S :

$$\begin{tikzcd} & S \arrow[dr] & \\ \hat{S} \arrow[ur, "\varepsilon"] \arrow[rr, "f"'] & & X \end{tikzcd}$$

Pf. the question is local on X. WMA X affine $X \hookrightarrow \mathbb{A}^n \rightsquigarrow f: \hat{S} \to \mathbb{A}^n$

$\rightsquigarrow f: \hat{S} \longrightarrow X = \mathbb{A}^1 \Rightarrow f$ defines a function on $S - \{p\}$ but $\forall$ function on $S - \{p\}$ extends to S.

For a surface S, define

$$B(S) := \left\{ \text{isom. classes of surfaces } S' \;\middle|\; \begin{array}{l} S' \to S \text{ bir. morph.} \\ S' \text{ birat'l eq. to } S \end{array} \right\}$$

If $S_1, S_2 \in B(S)$, then we say <u>S_1 dominate S_2</u> if $\exists$ a biration morphism $S_1 \longrightarrow S_2$ over S.

Can define an order on $B(S)$:

<u>Def</u> | a surface S is <u>minimal</u> if in $B(S)$, $[S]$ is minimal, that is $\forall$ birational morphism $S \to S'$ is an isom.

<u>Prop (Existence of a minimal model)</u>

every surface dominates a minimal surface.

<u>pf</u> let S be a suface, if S not minimal, then $\exists$ birat'l morph. $S \longrightarrow S_1$ which is not isom.

If S_1 not minimal, $\exists$ a birat'l non-isomorphic morphism

$$S_1 \longrightarrow S_2$$

repeat this process, then $\rho(S) > \rho(S_1) > \cdots$

While $\rho(S) \leq b_2(S)$

$\Rightarrow$ eventually get a minimal surface S_0 that is dominated by S.

$\square$