Birational maps of surfaces

Blow-ups

set-up : | S nonsingular projective surface $/\mathbb{C}$
| $p \in S$

Then $\exists$ a smooth surface $\hat{S}$ & a morph. $\varepsilon : \hat{S} \to \varepsilon$

Such that ① $\varepsilon|_{\varepsilon^{-1}(S-\{p\})} : \varepsilon^{-1}(S-\{p\}) \xrightarrow[\text{isom.}]{\cong} \varepsilon$

② $\varepsilon^{-1}(p) = E \cong \mathbb{P}^1$

ε is unique up to isom.

Call ε is the blow-up of S at p

E : exceptional curve of the blow-up.

Construction of blow-ups

Take a neighborhood $U \ni p$ with local coordinates x, y at p (i.e. the curves $x=0$, $y=0$ intersect transversely at $p=(0,0)$)

Can shrink U if necessary. We may assume p is the only point of U in the intersection $(x=0) \cap (y=0)$

define the subvariety $\underset{(x,y;[u:v])}{\hat{U}} \subset U \times \mathbb{P}^1_{[u:v]}$ by the equation

$$xv - yu = 0$$

$$\begin{tikzcd} \hat{U} \arrow[r, phantom, "\subset"] \arrow[dr, "\varepsilon"'] & U \times \mathbb{P}^1_{[u:v]} \arrow[d, "pr_2"] \\ & p \in \underset{(x,y)}{U} \subset \mathbb{A}^2 \end{tikzcd}$$

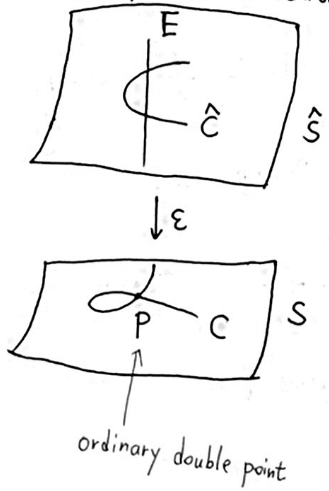
Observe that

- $\varepsilon : \hat{U} \to U$ isom. over points of U where at most one of the coordinates x, y vanishes.
  $(x,y;[y:x]) \longleftarrow (x,y)$
- $\varepsilon^{-1}(p) \cong \mathbb{P}^1$
  $\quad \| $
  $\{p\} \times \mathbb{P}^1$

$S = (S - \{p\}) \cup U \qquad (S - \{p\}) \cap U = U - \{p\}$

We get $\hat{S}$ by gluing $\hat{U}$ and $(S - \{p\})$ along $U - \{p\} \cong \hat{U} - \varepsilon^{-1}(p)$

Consider an irreducible curve C on S through p with multiplicity m. $\square$



☆ Points of E identified with the tangent directions on S at p

$p \in C = (f(x,y) = 0)$
$\|$
(x,y)

~~α-df~~

$p = (0,0)$

$f(x,y) = f_m(x,y) + f_{m+1} + \dots + f_n$, $\quad f_m(x,y) = \prod L_i^{r_i}$

L_i distinct lines called tangent lines to f at p

form of deg m

(if $\mathrm{mult}_p(C) = m$)

if $r_i = 1$, L_i called simple tangent
$r_i = 2$, double tangent.
if f has m distinct simple tangents at p,
Say p ordinary multiple pt

$$\begin{tikzcd} E \arrow[d] \arrow[r, phantom, "\subset"] & \hat{S} \arrow[d, "\varepsilon"] \\ p \arrow[r, phantom, "\in"] & S \end{tikzcd}$$

C : irreducible curve on S through p
with multiplicity $\mathrm{mult}_p(C) = m$

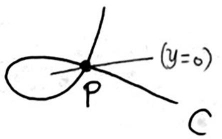
$$\hat{C} := \overline{\varepsilon^{-1}(C - \{p\})} \subset \hat{S}$$

$\uparrow$
irreducible curve, called strict transform of C
(proper transform)

<u>lemma</u> $\varepsilon^* C = \hat{C} + mE$

<u>pf</u> clearly $\varepsilon^* C = \hat{C} + kE$ for some $k \in \mathbb{Z}_{\geq 0}$.

Choose local coordinates x, y in a neighborhood U of p
such that the curve $(y=0)$ is not tangent to any branch of C at p.



Then in $\widehat{\mathcal{O}_{S,p}}$, the equation of C can be written as a formal power series

$$f = f_m(x,y) + f_{m+1}(x,y) + \cdots$$

where f_i : forms in x, y of degree i.

$m = \mathrm{mult}_p(C)$. $f_m(x,y) \neq 0$

$$\begin{tikzcd} (xv - yu = 0) \quad \hat{U} \arrow[r, hook] \arrow[dr, "\varepsilon"'] & U \times \mathbb{P}^1 \subset \mathbb{A}^2_{(x,y)} \times \mathbb{P}^1_{[u:v]} \arrow[d, "pr_1"] \\ & \underset{(x,y)}{U} \subset \mathbb{A}^2 \end{tikzcd}$$

$p \in U$, $p = (0,0)$

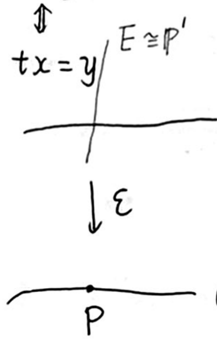
in a neighborhood of the point $(p, \infty) \in \hat{U} \subset \hat{S}$, $(p,\infty) = (p, [0:1])$
can take the functions x and $t = v/u$ as local coordinates.

then $\quad xv = yu \Leftrightarrow x\frac{v}{u} = y \Leftrightarrow \boxed{xt = y}$

$$\varepsilon^* f(x,y) = f(x, tx) \qquad f = f_m + f_{m+1} + \cdots$$

$$= x^m \underbrace{[f_m(1,t) + x f_{m+1}(1,t) + \cdots]}_{\substack{\uparrow \\ \text{defining } \hat{C}}}$$

$xv = yu$
$\Updownarrow$
$tx = y$



$(y=0) \Rightarrow tx = 0$
$t \in \mathbb{P}^1$

$(p, \infty) \in \hat{U}$ $\quad (x\overset{1}{v} - y\overset{0}{u} = 0)$

exceptional curve $E = \varepsilon^{-1}(p) \subset \hat{U}$
$\quad \|$
$(x = 0)$

$\Rightarrow k = m$

$$\Rightarrow \varepsilon^* C = \hat{C} + mE$$

$\square$

Prop. S surface

$\varepsilon : \hat{S} \longrightarrow S$ the blow-up of a point $p \in S$

$E \subset \hat{S}$ exceptional curve. Then

(1) $\exists$ an isomorphism $\operatorname{Pic} S \oplus \mathbb{Z} \xrightarrow{\sim} \operatorname{Pic} \hat{S}$

$$(D, n) \longmapsto \varepsilon^* D + nE$$

(2) D, D' divisors on S, then

$$\begin{cases} \varepsilon^* D . \varepsilon^* D' = D . D' \\ E . \varepsilon^* D = 0 \\ E^2 = -1 . \end{cases}$$

(3) $NS(\hat{S}) \cong NS(S) \oplus \mathbb{Z} \cdot [E]$

(4) $K_{\hat{S}} = \varepsilon^* K_S + E$.

<u>Pf</u>. (2) Recall that the intersection pairing is defined on Picard group, can replace D and D' by linearly equivalent divisors.

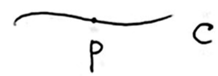
So we may assume that p doesn't lie on components of D & D'. Hence

$$\begin{cases} \varepsilon^* D . \varepsilon^* D' = D . D' & \leftarrow \varepsilon \text{ isom. outside } p \\ E . \varepsilon^* D = 0 & \leftarrow D \text{ not passes through } p \end{cases}$$

Choose a curve C passing through p with multiplicity 1.

$\Rightarrow$ the strict transform $\hat{C}$ meets E transversely at one point.

Corresponding to the tangent direction of C at p.



$$\left.\begin{matrix} \Rightarrow \hat{C} . E = 1 \\ \varepsilon^* C = \hat{C} + E \end{matrix}\right\} \Rightarrow \underbrace{\varepsilon^* C . E}_{=0} = \underbrace{\hat{C} . E}_{=1} + E^2$$

$$\Downarrow$$

$$E^2 = -1$$

(1). $\forall$ irreducible curve on $\hat{S}$ (other than E) is the strict transform of its image in S.

$\Rightarrow$ the map $\operatorname{Pic} S \oplus \mathbb{Z} \longrightarrow \operatorname{Pic} \hat{S}$

$$(D, n) \longmapsto \varepsilon^* D + nE$$

is surjective.

- Suppose that $\exists$ divisor $D \subset S$ such that $\varepsilon^* D + nE = 0$ ($\in \operatorname{Pic} \hat{S}$)

$\Rightarrow$ $\underbrace{(\varepsilon^* D + nE) E}_{= -n} = 0 \Rightarrow n = 0$

$\Rightarrow 0 = \varepsilon_* \varepsilon^* D = D \in \operatorname{Pic} S$

(3) note that ε_* & ε^* are defined on the Néron–Severi groups

&

$$\begin{tikzcd} \operatorname{Pic} S \times \operatorname{Pic} S \arrow[r, "\cdot"] \arrow[d, "c_1 \times c_1"'] \arrow[dr, phantom, "\circlearrowright"] & \mathbb{Z} \arrow[d, "\cong"] \\ NS(S) \times NS(S) \arrow[r, "\cup"', "\text{cup-product}"] & H^4(S, \mathbb{Z}) \end{tikzcd}$$

(4) Choose a meromorphic 2-form ω on S such that ω is holomorphic in a neighborhood of p & $\omega(p) \neq 0$.

It's clear that away from E the zeros and poles of $\varepsilon^*\omega$ are those of ω (via ε^*).

$$\Rightarrow \quad \operatorname{div}(\varepsilon^*\omega) = \varepsilon^* \operatorname{div}(\omega) + kE \text{ for some } k \in \mathbb{Z}.$$

i.e. $\varepsilon^* K_S + kE = K_{\hat{S}}$. By genus formula

$$\underset{\substack{\| \\ 0}}{g(E)} = 1 + \frac{1}{2}\Big(\underset{\substack{\| \\ -1}}{E^2} + \underset{\substack{\| \\ -k}}{K_{\hat{S}}.E}\Big) \Rightarrow k = 1.$$

Alternatively, if $\omega = dx \wedge dy$ where x, y local coordinates at $p \in S$

$y = tx$ then $\varepsilon^*\omega = dx \wedge d(tx) = x\, dx \wedge dt$ in local coordinates x, t at a point of $E \subset \hat{S}$

$\Rightarrow \varepsilon^* K_S + E = K_{\hat{S}}$

Rational maps ~~& linear systems~~

~~rat'l maps~~

Set-up X, Y Varieties with X irreducible

a rational map $\phi : X \dashrightarrow Y$ is a morphism $U \to Y$, $U \subset X$ open, which cannot be extended to any larger open subset.

We say that ϕ is defined at x if $x \in U$.

Suppose that S is a smooth surface & $\varphi : S \dashrightarrow Y$ rat'l map then the undefined set of φ, $\Sigma := S - U$, is a finite set. (called indeterminancy locus of φ)

Prop X normal Variety (e.g. smooth), Y projective variety, $\varphi : X \dashrightarrow Y$ a rational map then the indeterminacy locus of φ has codim ≥ 2.

Pf $X \dashrightarrow Y \hookrightarrow \mathbb{P}^n$

We can reduce to the case $Y = \mathbb{P}^n$

Now consider rational map $\varphi : X \dashrightarrow \mathbb{P}^n$

The question is local.

for $\forall$ point x in the indeterminacy locus of φ,

X normal $\Rightarrow \mathcal{O}_{x,X}$ integrally closed domain

For any codim 1 component Z of indeterminacy locus of φ passing through x, then $\mathcal{O}_{X,Z}$ is a DVR, say Z is defined by a single equation $g \in \mathcal{O}_{X,x}$.

$\varphi : X \dashrightarrow \mathbb{P}^n$ given by $(\varphi_0, \varphi_1, \cdots, \varphi_n)$ with $\varphi_i \in k(X)$ rational function

Can multiply by a common factor (in $k(X)$) such that these φ_i no common factor & $\varphi_i \in \mathcal{O}_{X,x}$

$\Rightarrow$ the indeterminacy locus of φ in a neighborhood of x is the common zero lcus

$$\bigcap_{i=0}^{n} (\varphi_i = 0).$$

$\Rightarrow$ g is a common factor of these φ_i, contradicting to the choice of the φ_i. $\square$

In particular,

$\varphi : C \dashrightarrow \mathbb{P}^n$ rational map. C smooth curve $\Rightarrow$ φ is a morphism

$\varphi : S \dashrightarrow \mathbb{P}^n$ rational map. S smooth surface $\Rightarrow$ indeterminacy locus of φ is a finite set of points of S

Now let $\varphi : S \dashrightarrow Y$ be a rational map. where

S a smooth surface & Y projective variety. &
Σ the indeterminacy locus of φ

If $C \subset S$ an irreducible curve, then φ defined on $C - \Sigma$

In this case, the image of C under φ defined to be

$$\varphi(C) := \overline{\varphi(C - \Sigma)} \subset Y$$

taking closure

Similarly, $\varphi(S) := \overline{\varphi(S - \Sigma)} \subset Y$

Note that $\forall$ codim 2 subset does not affect the Picard groups

that is, $\operatorname{Pic} S \xrightarrow[\text{restr.}]{\cong} \operatorname{Pic}(S - \Sigma)$

$\varphi : S - \Sigma \longrightarrow Y \rightsquigarrow \operatorname{Pic} Y \xrightarrow{\varphi^*} \operatorname{Pic}(S - \Sigma)$ $\Bigg\} \Rightarrow$

$$\begin{tikzcd} \operatorname{Pic} Y \arrow[r] \arrow[rr, bend right, "\text{still denote}", "\varphi^*"'] & \operatorname{Pic}(S-\Sigma) \arrow[r, "\cong"] & \operatorname{Pic} S \end{tikzcd}$$

linear systems.

<u>set-up</u>

S : surface

D: divisor on S, say $D = \sum n_i C_i$

let

$$|D| := \left\{ D' \geq 0 \;\middle|\; \begin{array}{l} D' \text{ effective divisor on } S \\ \text{with } D \underset{lin}{\sim} D' \end{array} \right\}$$

Called the linear system associated to D. By definition,

for $\forall D' \in |D|$, $\exists$ a rational function $0 \neq f \in k(S)$ s.t.

$$D' = D + \operatorname{div}(f)$$

Such a section $f \in k(S)$ determined uniquely up to a scalar

$\Rightarrow$ if we consider

$$L(D) := \{ f \in k(S)^* \mid \operatorname{div}(f) + D \geq 0 \} \cup \{0\}$$

then can identify

$$|D| \cong \mathbb{P}(L(D))$$

<u>Rmk</u> $L(D)$ is a vector space which is the set of all rational Sections/functions of S having order $\geq -n_i$ along C_i.

For $\forall_a$ $f_0 \in H^0(S, \mathcal{O}_S(D))$ with $\operatorname{div}(f_0) = D$, then

for $\forall f \in H^0(S, \mathcal{O}_S(D))$, the quotient $t_f := f/f_0 \in k(S)$

with $\operatorname{div}(t_f) = \operatorname{div}(f) - \operatorname{div}(f_0) \geq -D$, i.e. $t_f \in L(D)$

& $\operatorname{div}(f) = D + \operatorname{div}(t_f) \geq 0$

$$\begin{array}{c} \operatorname{div}(f) \\ \cap \\ |D| \end{array}$$

Conversely, for $\forall t_f \in L(D)$

$$s := t_f \cdot f_0 \in H^0(S, \mathcal{O}_S(D))$$

$\Rightarrow$ We have an identification

$$L(D) \xrightarrow{\otimes f_0} H^0(S, \mathcal{O}_S(D))$$

<u>Summary</u>

$$|D| \cong \mathbb{P}(L(D)) \cong \left(H^0(S, \mathcal{O}_S(D)) - \{0\}\right) / \mathbb{C}^*$$

a linear subspace $P \subset |D|$ called a linear (sub-)system

$\updownarrow$ corresp.

a Subvector space $V_P \subset H^0(\mathcal{O}_S(D))$

We say the linear system P is complete if $P = |D|$

$\dim |P| := \dim_{\mathbb{C}} \mathbb{P}(V_P)$

linear systems of dim 1, 2, or 3 called pencils, nets or webs, respectively.

let P be a linear system on S, a curve C is called a <u>fixed component</u> of P if for $\forall$ divisor $D \in P$, $C \subseteq D$.

The <u>fixed part</u> of P is the biggest divisor Z with $Z \subseteq D$ for $\forall D \in P$

A point $x \in S$ called a <u>base point</u> of P if for $\forall D \in P$ $x \in D$

Collecting all base points of P, define the <u>base locus</u> of P as

$$Bs(P) := \{x \in S \mid x \in \operatorname{Supp} D, \text{ for } \forall D \in P\}$$

For surface S and linear system P on S, let Z be the fixed part of P (if any), then $P - Z$ is a linear system M having no fixed part & only a finite number of base points.

i.e. $P = \underset{\nwarrow}{M} + Z$
moving/mobile part of P

Clearly, in this case

$$\#\{\text{base points of } M\} \leqslant D^2 \quad \text{for } D \in \overset{\overset{\text{no fixed part.}}{\downarrow}}{M}$$

<u>Bertini's Theorem</u>

> Let P be a linear system on a smooth variety X & $D \in P$ a general member, then D is smooth outside $Bs(P)$.

<u>Pf</u> If the generic element of P is singular away from $Bs(P)$ then the generic element of a generic pencil $\subseteq P$ will be singular away from $Bs(P)$

$\Rightarrow$ suffices to prove Bertini for a pencil $\{D_\lambda\}$

The question is local. We may assume locally general member

$$D := D_\lambda = \{f(x_1, x_2, \dots, x_n) + \lambda\, g(x_1, x_2, \dots, x_n) = 0\}$$

$0 \in \operatorname{Supp} D_\lambda$ is a singular point & $0 \notin Bs|D_\lambda|$

then $f(0) \neq 0$ (if $f(0) = 0$, $0 \in D \Rightarrow g(0) = 0 \Rightarrow 0 \in Bs|D_\lambda|$)

$\Rightarrow g(0) \neq 0 \Rightarrow \lambda = -\frac{f(0)}{g(0)}$

D singular at $0 \Leftrightarrow \left(\frac{\partial f}{\partial x_i} - \lambda \frac{\partial g}{\partial x_i}\right)\Big|_{x=0} = 0$ for $\forall i \Rightarrow \frac{\partial}{\partial x_i}\left(\frac{f}{g}\right)\Big|_{x=0} = 0 \quad \forall i$

$\Rightarrow \frac{f}{g}$ constant on $\forall$ <u>connected component</u> (finitely many) of singular locus D_{sing} outside $Bs|D| \Rightarrow \exists$ finitely many divisors meeting D_{sing}