Curves on surfaces

Some standard exact sequences

S : smooth projective surface $/\mathbb{C}$

a curve C on S is a 1-dim'l closed subvariety of S locally defined by one equation $(f=0)$

$$C : \left(f = c \cdot \prod f_i^{e_i} = 0\right) \longleftrightarrow C = \bigcup e_i C_i$$

(f_i irreducible)

$$\text{Curves} \xleftrightarrow{1-1} \text{effective divisors on } S$$

let $\iota : C \hookrightarrow S$ closed immersion, then $\iota^\# : \mathcal{O}_S \twoheadrightarrow \iota_* \mathcal{O}_C$

denote by $\mathcal{I}_C$ the ideal sheaf of C in S $\qquad \mathcal{O}_C$

$\rightsquigarrow$ the natural sequence

$$0 \to \mathcal{I}_C \to \mathcal{O}_S \to \mathcal{O}_C \to 0 \quad \text{exact}$$

For any divisor D on S, the corresponding invertible sheaf $\mathcal{O}_S(D)$

$$\mathcal{O}_C(D) := \iota^* \mathcal{O}_S(D) = \mathcal{O}_S(D)|_C$$

$\uparrow$ restriction of $\mathcal{O}_S(D)$ to C

<u>Fact</u> for effective divisor $E \geq 0$, $\mathcal{I}_E \cong \mathcal{O}_S(-E)$

i.e. $0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0$ exact

twisted by $\mathcal{O}_S(C)$, we have exact seq

$$0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \iota_* \mathcal{O}_C \otimes \mathcal{O}_S(C) \to 0$$

$\wr\wr$ Projection formula

$$\iota_*(\mathcal{O}_C \otimes \iota^* \mathcal{O}_S(C))$$

$$\| $$

$$\mathcal{O}_S(C)|_C = \mathcal{O}_C(C)$$

- If C smooth, then we have conormal bundle sequence

$$\begin{array}{ccccccc} 0 \to & \mathcal{N}_{C/S}^\vee & \longrightarrow & \Omega_S^1 \otimes \mathcal{O}_C & \longrightarrow & \Omega_C^1 & \to 0 \\ & \| & & \| & & \uparrow & \\ & \mathcal{I}_C/\mathcal{I}_C^2 & & \Omega_S^1|_C & & \text{locally free of rk 1} & \end{array}$$

taking dual, normal bundle sequence

$$0 \to \mathcal{T}_C \longrightarrow \mathcal{T}_S|_C \longrightarrow \mathcal{N}_{C/S} \to 0$$

- If C non-reduced, say $C \cancel{A} = A + B$ a sum of effective divisors A, B

Note that $\mathcal{I}_C \subset \mathcal{I}_B \subset \mathcal{O}_S$

We have the following commutative diagram by five lemma / Snake lemma

$$
\begin{tikzcd}
 & 0 \arrow[d] & & 0 \arrow[d] & \\
 & & & \mathcal{I}_B/\mathcal{I}_C \arrow[d] & \\
0 \arrow[r] & \mathcal{I}_C = \mathcal{O}_S(-C) \arrow[r] \arrow[d, hook] & \mathcal{O}_S \arrow[r] \arrow[d, equal] & \mathcal{O}_C \arrow[r] \arrow[d] & 0 \\
0 \arrow[r] & \mathcal{I}_B = \mathcal{O}_S(-B) \arrow[r] \arrow[d] & \mathcal{O}_S \arrow[r] & \mathcal{O}_B \arrow[r] \arrow[d] & 0 \\
 & \mathcal{O}_A(-B) \arrow[d] & & 0 & \\
 & 0 & & &
\end{tikzcd}
$$

$0 \to \mathcal{O}_S(-A) \to \mathcal{O}_S \to \mathcal{O}_A \to 0$

twisted by $\mathcal{O}_S(-B)$

$$0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S(-B) \to \mathcal{O}_A(-B) \to 0$$

$\Rightarrow$ $\mathcal{I}_B/\mathcal{I}_C \cong \mathcal{O}_A(-B)$ $\leftarrow$ a line bundle on A

$\Rightarrow$ $0 \to \mathcal{O}_A(-B) \to \mathcal{O}_C \xrightarrow{\text{restr.}} \mathcal{O}_B \to 0$ exact

Called the decomposition sequence for $C = A + B$

Special case : $2C = C + C$, then $\mathcal{I}_C/\mathcal{I}_C^2 \cong \mathcal{O}_C(-C)$

Called conormal bundle of C in S

(even in the non-smooth case)

• If $C \subset S$ reduced, let $\nu : \tilde{C} \to C$ be the normalization of C, then there is a normalization sequence

$$0 \to \mathcal{O}_C \to \nu_* \mathcal{O}_{\tilde{C}} \to \delta \to 0$$

Supported on singular points of C

For smooth curve $C \subset S$, we have the adjunction formula

$$\cancel{\omega_C =} \; K_C \cong K_S \otimes \mathcal{O}_C(C) = \mathcal{O}_S(K_S + C)|_C$$

For singular curve $C \subset S$,

$$\omega_C = K_S \otimes \mathcal{O}_C(C) \quad \text{dualizing sheaf of } C$$

From the exact seq. $0 \to \mathcal{O}_S \to \mathcal{O}_S(C) \to \mathcal{O}_C(C) \to 0$

twisted by K_S

$$0 \to K_S \to K_S \otimes \mathcal{O}_S(C) \xrightarrow{r} \omega_C \to 0$$

dualizing sheaf

Picard group of an embedded curve

If C smooth curve, then $\exists$ exponential exact sequence

$$0 \to \mathbb{Z}_C \longrightarrow \mathcal{O}_C \xrightarrow{e} \mathcal{O}_C^* \longrightarrow 0$$

It exists even for any curve (reduced or not).

Taking cohomology, (exponential cohomology sequence)

$$\to H^1(C,\mathbb{Z}) \to H^1(\mathcal{O}_C) \longrightarrow \underset{\substack{\| \\ \mathrm{Pic}(C)}}{H^1(\mathcal{O}_C^*)} \xrightarrow{c_1} H^2(C,\mathbb{Z}) \to 0$$

Prop | If $C \subset S$ a curve, then $H^1(C,\mathbb{Z})$ is a discrete subgroup of $H^1(\mathcal{O}_C)$. So $\mathrm{Pic}^0 C = H^1(\mathcal{O}_C)/H^1(C,\mathbb{Z})$ carries the structure of a (Hausdorff) abelian complex Lie group.

<u>pf</u>.

$$\begin{tikzcd}
H^1(C,\mathbb{Z}) \arrow[r] \arrow[d, equal] & H^1(\mathcal{O}_C) \arrow[d, "\text{restr.}"] \\
H^1(C_{red},\mathbb{Z}) \arrow[r] & H^1(\mathcal{O}_{C_{red}})
\end{tikzcd}$$

the assertion holds for C if it holds for C_{red}.

So we may assume C reduced.

Note that $\mathcal{O} \xrightarrow{e} \mathcal{O}^*$ is (stalkwise) surjective

$\Rightarrow H^0(C,\mathcal{O}_C) \to H^0(C,\mathcal{O}_C^*)$ surjective & hence

$$H^1(C,\mathbb{Z}) \hookrightarrow H^1(\mathcal{O}_C)$$

- To prove that the image of $H^1(C,\mathbb{Z})$ is discrete in $H^1(\mathcal{O}_C)$

Consider the normalization sequence for C

$$\begin{tikzcd}
0 \arrow[r] & \mathcal{O}_C \arrow[r] & \nu_* \mathcal{O}_{\tilde{C}} \arrow[r] & \delta \arrow[r] & 0 \\
0 \arrow[r] & \mathbb{Z}_C \arrow[r] \arrow[u] & \nu_* \mathbb{Z}_{\tilde{C}} \arrow[r] \arrow[u] & \Sigma \arrow[r] \arrow[u] & 0
\end{tikzcd}$$

taking cohomology, it induces

$$\begin{tikzcd}
H^0(C,\Sigma) \arrow[r] \arrow[d] & H^1(C,\mathbb{Z}) \arrow[r, "\nu^*"] \arrow[d, hook] & H^1(\tilde{C},\mathbb{Z}) \arrow[d, hook] \\
H^0(C,\delta) \arrow[r] & H^1(C,\mathcal{O}_C) \arrow[r, "\nu^*"] & H^1(\mathcal{O}_{\tilde{C}})
\end{tikzcd}$$

$\tilde{C}$ nonsingular $\Rightarrow H^1(\tilde{C},\mathbb{Z}) \subset H^1(\mathcal{O}_{\tilde{C}})$ discrete

suffices to show that $H^0(C,\Sigma) \to H^0(C,\delta)$ has discrete image.

(Check it locally)

If $y \in C$ a singular point $\nu^{-1}(y) = \{x_1, \cdots, x_r\}$

$U \subset C$: a small neighborhood of x.

$\nu^{-1}(U) = U_1 \cup \cdots \cup U_r \subset \tilde{C}$.

Consider the diagram

$$\begin{array}{ccccccccc} 0 \to & H^0(U, \mathbb{Z}) & \xrightarrow{\nu^*} & \bigoplus_{i=1}^r H^0(U_i, \mathbb{Z}) & \to & \Sigma_y & \to 0 \\ & \downarrow & & \downarrow & & \downarrow & \\ 0 \to & H^0(\mathcal{O}_U) & \to & \bigoplus_{i=1}^r H^0(\mathcal{O}_{U_i}) & \to & \delta_y & \to 0 \end{array}$$

If $e_i \in H^0(U_i, \mathbb{Z})$ denotes the constant 1, then

$$e_1, \cdots, e_r \text{ generate } \bigoplus_{i=1}^r H^0(U_i, \mathbb{Z}).$$

The images of these e_i in $\bigoplus H^0(U_i, \mathcal{O}_{U_i})$ linearly independent$/\mathbb{C}$

& span over $\mathbb{R}$ a vector space V of dim r.

The intersection

$V_0 := V \cap H^0(\mathcal{O}_U)$ is the 1-dim'l $\mathbb{R}$-vector space spanned by $1 \in H^0(\mathcal{O}_U)$.

The image of Σ_y in δ_y is a group of rank $r-1$ spanning the $(r-1)$-dim'l $\mathbb{R}$-vector space V/V_0

$\Rightarrow$ the image of Σ_y is a lattice in $V/V_0 \subset \delta_y$.

& hence discrete in δ_y. $\square$

$\nu : \tilde{C} \to C$ normalization

- If C reduced curve, $\tilde{C} = \bigsqcup \tilde{C}_i$, say $C = C_1 \cup \cdots \cup C_r$ union of distinct irred. comp.

then ν^* induces an isom.

$$H^2(C, \mathbb{Z}) \underset{\sim}{\xrightarrow{\nu^*}} \bigoplus H^2(\tilde{C}_i, \mathbb{Z}) \cong \mathbb{Z}^{\oplus r}$$
$$\uparrow c_1$$
$$\mathrm{Pic}\, C$$

If $L \in \mathrm{Pic}\, C$, then $c_1(L) \in H^2(C, \mathbb{Z})$
$\quad\quad \| \leftarrow$ via ν^*
$\quad\quad (\ell_1, \cdots, \ell_r) \in \mathbb{Z}^r$

then define the degree of L

$$\deg L := \ell_1 + \cdots + \ell_r$$

- If C non-reduced curve, say $C = n_1 C_1 + \cdots + n_r C_r$

Put $\deg L := n_1 \ell_1 + \cdots + n_r \ell_r$ with $\ell_i = \deg(L|_{(C_i)_{red}})$

- For locally free sheaf $\mathcal{F}$, define

$$\deg(\mathcal{F}) := \deg(\det \mathcal{F})$$

Riemann-Roch for embedded curves.

Riemann-Roch theorem for embedded curves

$C \subset S$ a curve, $\mathcal{E}$: locally free rank r $\mathcal{O}_C$-sheaf

then $\chi(\mathcal{E}) = \deg(\mathcal{E}) + r \cdot \chi(\mathcal{O}_C)$

proof

Case 1 C smooth

Case 1.1 assume $\mathcal{E}$ is a line bundle, say $\mathcal{E} = \mathcal{O}_C(D)$ for some divisor D on C

If $D=0$. WTS $h^0(\mathcal{O}_C) - h^1(\mathcal{O}_C) = 1 - g(C)$

note that $H^0(C, \mathcal{O}_C) = \mathbb{C}$

$H^1(C, \mathcal{O}_C) \cong H^0(C, K_C) \leftarrow g$-dim'l

For arbitrary divisor D. let P be any point & viewed as a closed subscheme of C with the structure sheaf $\mathcal{O}_P := k(P)$ sky scraper sheaf at P

Consider the structure exact seq.

$$0 \to \mathcal{O}_C(-P) \to \mathcal{O}_C \to k(P) \to 0$$

twisted by $\mathcal{O}_C(D+P)$, one obtains

$$0 \to \mathcal{O}_C(D) \to \mathcal{O}_C(D+P) \to k(P) \to 0$$

then $\chi(\mathcal{O}_C(D+P)) = \chi(\mathcal{O}_C(D)) + \underbrace{\chi(k(P))}_{\substack{\| \\ h^0(k(P))=1}}$

$\deg \mathcal{O}_C(D+P) = \deg \mathcal{O}_C(D) + 1$

$\Rightarrow$ R.R. formula holds for $D+P \Leftrightarrow$ it holds for D

$\Rightarrow$ R.R. formula holds for D, since it holds for $D=0$.

Case 1.2 $\mathcal{E}$ locally free sheaf of rank r on C

Chose a satured invertible subsheaf $\mathcal{E}_1 \subset \mathcal{E}$, Consider the short exact seq.

$$0 \to \mathcal{E}_1 \to \mathcal{E} \to \underset{\substack{\| \\ \mathcal{E}/\mathcal{E}_1}}{\mathcal{E}_2} \to 0$$

$\mathcal{E}/\mathcal{E}_1 \leftarrow$ locally free of rk $r-1$

Using induction on rank $\mathcal{E}$

□

Case 2 C reduced. Say $C = C_1 \cup \cdots \cup C_m$

Consider the normalization $\nu : \tilde{C} \to C$ of C

Put $\tilde{\mathcal{E}} := \nu^* \mathcal{E}$

by Leray spectral sequence $\begin{cases} E_2^{pq} = H^p(R^q\nu_*\tilde{\mathcal{E}}) \Rightarrow H^{p+q}(\tilde{\mathcal{E}}) \\ E_2^{pq} = H^p(R^q\nu_*\mathcal{O}_{\tilde{C}}) \Rightarrow H^{p+q}(\mathcal{O}_{\tilde{C}}) \end{cases}$

one has $h^1(\nu_*\tilde{\mathcal{E}}) = h^1(\tilde{\mathcal{E}})$ & $h^1(\nu_*\mathcal{O}_{\tilde{C}}) = h^1(\mathcal{O}_{\tilde{C}})$

$\Rightarrow \chi(\nu_*\tilde{\mathcal{E}}) = \chi(\tilde{\mathcal{E}})$ & $\chi(\nu_*\mathcal{O}_{\tilde{C}}) = \chi(\mathcal{O}_{\tilde{C}})$

$(0 \to \mathcal{O}_C \to \nu_*\mathcal{O}_{\tilde{C}} \to \underset{\substack{\uparrow \\ \text{supported on singular pts}}}{\delta} \to 0 \quad \Rightarrow \quad 0 \to \mathcal{E} \to \nu_*\tilde{\mathcal{E}} \to \delta' \to 0$

$\begin{cases} \dim H^0(C,\delta) = \sum_{\substack{x\in C \\ \text{singular}}} \dim \delta_x \\ \dim H^1(C,\delta) = 0 \end{cases}$

$\Rightarrow \underline{\deg(\tilde{\mathcal{E}}) = \deg(\mathcal{E})}$

$H^0(C,\delta) = \bigoplus_{\substack{x\in C \\ \text{singular}}} \delta_x$

Applying R.-R. on $\tilde{C}$,

$$\begin{array}{ccccc} \chi(\tilde{\mathcal{E}}) & = & \deg\tilde{\mathcal{E}} & + & r.\chi(\mathcal{O}_{\tilde{C}}) \\ \| & & \| & & \| \\ \chi(\nu_*\tilde{\mathcal{E}}) & & \deg(\mathcal{E}) & & r.\chi(\nu_*\mathcal{O}_{\tilde{C}}) \end{array}$$

Since $\mathcal{E}$ locally free,

$$(\nu_*\tilde{\mathcal{E}})/\mathcal{E} \cong r.\left(\nu_*\mathcal{O}_{\tilde{C}}/\mathcal{O}_C\right)$$

$$\begin{aligned} \Rightarrow \chi(\mathcal{E}) &= \chi(\nu_*\tilde{\mathcal{E}}) - \chi(\nu_*\tilde{\mathcal{E}}/\mathcal{E}) \\ &= \underline{\chi(\tilde{\mathcal{E}})} - r\chi(\nu_*\mathcal{O}_{\tilde{C}}/\mathcal{O}_C) \\ &= \deg\mathcal{E} + r\chi(\nu_*\mathcal{O}_{\tilde{C}}) - r\chi(\nu_*\mathcal{O}_{\tilde{C}}/\mathcal{O}_C) \\ &= \deg\mathcal{E} + r\chi(\mathcal{O}_C) \end{aligned}$$

$\square$

Case 3 C not reduced.

Consider the decomposition sequence for $C = A+B \begin{cases} A \geq 0 \\ B \geq 0 \end{cases}$

$$0 \to \mathcal{O}_A(-B) \to \mathcal{O}_C \xrightarrow{\text{restr.}} \mathcal{O}_B \to 0$$

twisted by $\otimes\mathcal{E}$

$$0 \to \mathcal{E}|_A \otimes \mathcal{O}_A(-B) \to \mathcal{E} \to \mathcal{E}|_B \to 0$$

$$\begin{aligned} \Rightarrow \chi(\mathcal{O}_C) &= \chi(\mathcal{O}_B) + \chi(\mathcal{O}_A(-B)) \\ \chi(\mathcal{E}) &= \chi(\mathcal{E}|_B) + \chi(\mathcal{E}|_A \otimes \mathcal{O}_A(-B)) \end{aligned}$$

Applying R.-R. on A & B,

$$\chi(\mathcal{E}|_B) = \deg\mathcal{E}|_B + r\chi(\mathcal{O}_B)$$

$\deg \mathcal{E}|_A + r.\deg\mathcal{O}_A(-B)$

$$\begin{aligned} \chi(\mathcal{E}|_A \otimes \mathcal{O}_A(-B)) &= \deg(\mathcal{E}|_A \otimes \mathcal{O}_A(-B)) + r\chi(\mathcal{O}_A) \\ &= \deg\mathcal{E}|_A + r\chi(\mathcal{O}_A(-B)) \end{aligned}$$

by def, $\deg\mathcal{E} = \deg(\mathcal{E}|_A) + \deg(\mathcal{E}|_B)$. By above equations, R-R holds for $\mathcal{E}$ on C.

Serre duality for embedded curves.

<u>Thm</u>

S : smooth projective surface $/\mathbb{C}$

$C \subset S$ curve (not necessarily reduced)

then $\exists$ an epimorphism $\mathrm{tr} : H^1(\omega_C) \longrightarrow \mathbb{C}$ such that

the cup-product pairing

$$H^1(\mathcal{E}) \otimes H^0(\mathcal{E}^\vee \otimes \omega_C) \xrightarrow{\cup} H^1(\omega_C) \xrightarrow{\mathrm{tr}} \mathbb{C}$$

defined for any $\mathcal{O}_C$-sheaf $\mathcal{E}$.

If $\mathcal{E}$ locally free $\mathcal{O}_C$-sheaf, then it is perfect.

here the trace map $\mathrm{tr} : H^1(\omega_C) \longrightarrow \mathbb{C}$ exists

In C smooth case (classical Serre duality)

C reduced projective (Serre)

general projective schemes (Grothendieck)

Intersection multiplicities / numbers

<u>Setting</u> S : smooth projective surface $/\mathbb{C}$

C, C' : two distinct irreducible curves on S

$x \in C \cap C'$

If $f, g \in \mathcal{O}_{S,x}$ are local equations for C, C', resp.

<u>Def</u> define the intersection multiplicity of C and C' at x to be

$$m_x(C \cap C') := \dim_{\mathbb{C}} \frac{\mathcal{O}_{S,x}}{(f,g)}$$

← finite-dim'l $\mathbb{C}$-vector space

$m_x(C \cap C') = 1$

$\Updownarrow$

f and g generate the maximal ideal $\mathfrak{m}_x \subset \mathcal{O}_{S,x}$

i.e. f, g form a system of local coordinates in a neighborhood of x

in this case, C and C' said to be <u>transverse at x</u>.

<u>Def</u> If C, C' two distinct irreducible curves on S

then the intersection number $C.C'$ defined by

$$C.C' := \sum_{x \in C \cap C'} m_x(C \cap C')$$

Recall that the ideal sheaf $\mathcal{I}_C$ defining C is just the invertible sheaf $\mathcal{O}_S(-C)$.

define $$\mathcal{O}_{C\cap C'} := \frac{\mathcal{O}_S}{(\mathcal{O}_S(-C) + \mathcal{O}_S(-C'))}$$

it is a skyscraper sheaf concentrated at the finite set $C \cap C'$

for $\forall\, x \in C \cap C'$,

$$(\mathcal{O}_{C\cap C'})_x = \mathcal{O}_{x,S} / (f,g)$$

$$\Longrightarrow C.C' = \dim H^0(S, \mathcal{O}_{C\cap C'})$$

<u>THEOREM</u>

For $L, L' \in \operatorname{Pic} S$, define

$$(-.-) : \operatorname{Pic} S \times \operatorname{Pic} S \longrightarrow \mathbb{Z}$$
$$(L, L') \longmapsto L.L'$$

$$L.L' := \chi(\mathcal{O}_S) - \chi(L^{-1}) - \chi(L'^{-1}) + \chi(L^{-1} \otimes L'^{-1})$$

then $(-.-)$ is a symmetric bilinear form on $\operatorname{Pic} S$ such that if C, C' two distinct irred. curves then $\mathcal{O}_S(C).\mathcal{O}_S(C') = C.C'$ (intersection number)

Observation 1

Suppose that $s \in H^0(\mathcal{O}_S(C))$ a nonzero section vanishing on C
(resp. $s' \in H^0(\mathcal{O}_S(C'))$ (resp. C')

then the sequence

$$0 \to \mathcal{O}_S(-C-C') \xrightarrow{(s',-s)} \mathcal{O}_S(-C) \oplus \mathcal{O}_S(-C') \xrightarrow{(s,s')} \mathcal{O}_S \to \mathcal{O}_{C \cap C'} \to 0$$

is exact.

Pf $f, g \in \mathcal{O}_{x,S}$ local equations for C, C' at $x \in S$

suffices to show the seq.

$$0 \to \mathcal{O}_x \xrightarrow{(g,-f)} \mathcal{O}_x^{\oplus 2} \xrightarrow{(f,g)} \mathcal{O}_x \to \mathcal{O}_x/(f,g) \to 0$$

$(t_1, t_2) \longmapsto ft_1 + gt_2$

$t \longmapsto (gt, -ft)$

is exact. i.e. suffices to check the exactness at $\mathcal{O}_x^{\oplus 2}$.

$\Leftrightarrow$

($\ker(f,g) \subset \operatorname{Im}(g,-f)$

if $(a,b) \in \mathcal{O}_x^{\oplus 2}$ with $af + bg = 0 \in \mathcal{O}_x$ then $\exists k \in \mathcal{O}_x$ s.t. $\begin{cases} gk = a \\ -fk = b \end{cases}$)

In fact, $\mathcal{O}_{x,S}$ is a UFD & f, g are coprime
(otherwise C and C' would have a common component!)

then the desired above result holds.

Observation 2

$\mathcal{O}_S(C) . \mathcal{O}_S(C') = C . C'$

by definition, $\mathcal{O}_S(C) . \mathcal{O}_S(C') = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(-C)) - \chi(\mathcal{O}_S(-C')) + \chi(\mathcal{O}_S(-C-C'))$

additivity of Euler–Poincaré characteristic

$$\begin{aligned} &= \chi(\mathcal{O}_{C \cap C'}) \\ &= h^0(\mathcal{O}_{C \cap C'}) \\ &= C . C' \end{aligned}$$

Observation 3

$(- . -)$ is symmetric form on $\operatorname{Pic} S$ by def.

Observation 4

C : smooth irreducible curve on S
$L \in \operatorname{Pic} S$
then $\mathcal{O}_S(C) . L = \deg(L|_C)$

Pf Consider the structure sequence

$$0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0$$

twisted by L^{-1}

$$0 \to \underbrace{\mathcal{O}_S(-C) \otimes L^{-1}}_{= L^{-1}(-C)} \to L^{-1} \to \underbrace{L^{-1} \otimes \mathcal{O}_C}_{= L^{-1}|_C} \to 0$$

$$\Rightarrow \left| \begin{array}{l} \chi(\mathcal{O}_S) = \chi(\mathcal{O}_S(-C)) + \chi(\mathcal{O}_C) \\ \chi(L^{-1}) = \chi(L^{-1}(-C)) + \chi(L^{-1}|_C) \end{array} \right. \quad (*)$$

by def.

$$\mathcal{O}_S(C).L = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(-C)) - \chi(L^{-1}) + \chi(L^{-1}(-C))$$

$$\overset{(*)}{=} \chi(\mathcal{O}_C) - \chi(L^{-1}|_C)$$

$$\overset{\text{R.R.}}{=} -\deg(L^{-1}|_C)$$

$$= \deg(L|_C)$$

$\square$

<u>Observation 5</u> $(-.-)$ is a bilinear form on $\mathrm{Pic}\, S$.

For $L_1, L_2, L_3 \in \mathrm{Pic}\, S$, consider the expression

$$s(L_1, L_2, L_3) := L_1.(L_2 \otimes L_3) - L_1.L_2 - L_1.L_3$$

- by def. $s(L_1, L_2, L_3)$ is symmetric in L_1, L_2, L_3;
- if C smooth irreducible curve & $L_1 = \mathcal{O}_S(C)$, then

$$s(L_1, L_2, L_3) = \mathcal{O}_S(C).(L_2 \otimes L_3) - \mathcal{O}_S(C).L_2 - \mathcal{O}_S(C).L_3$$

$$\overset{\text{Observation 4}}{=} \deg(L_2 \otimes L_3)|_C - \deg(L_2|_C) - \deg(L_3|_C) = 0$$

by symmetry, $s(L_1, L_2, L_3) = 0$ if L_2 or L_3 is of this form.

- let L, L' be any two invertible sheaves.

<u>Can write</u> $L' = \mathcal{O}_S(A - B)$ where A, B two smooth curves

then $s(L, L', \mathcal{O}_S(B)) = 0$.

$$\Rightarrow L.L' = L.\mathcal{O}_S(A) - L.\mathcal{O}_S(B)$$

$$\overset{\text{ob.4}}{=} \deg(L|_A) - \deg(L|_B)$$

i.e. $L.L'$ is linear in L
by symmetry of $L.L'$
$\Big\} \Rightarrow L.L'$ bilinear in L, L'.

If D, D' two divisors on S, write $D.D'$ for $\mathcal{O}_S(D).\mathcal{O}_S(D')$

only depend on D, D' up to linearly equiv.

<u>Prop</u> ① If C smooth curve, $f: S \to C$ surjective morphism, F a fibre of f.

then $F^2 = 0$

② If T surface, $g: S \to T$ generically finite morph. of deg d, D, D' two divisors on T

then $g^*D . g^*D' = d\,(D.D')$

<u>Pf.</u> ① Say $F = f^*[x]$ for some $x \in C$.

There exists a divisor A on C with $A \underset{\text{lin}}{\sim} x$ & $x \notin A$.

$$\Rightarrow F \underset{\text{lin.}}{\sim} f^*A$$

f^*A linear combination of fibres of f

all distinct from F

$\Rightarrow$ $F^2 = F.f^*A = 0$

intersection numbers defined up to linear equivalence.

② D, D' divisor on S, H a hyperplane section of S (w.r.t. an embedding of S)

Serre's thm. $\rightsquigarrow$ $\exists\, n, m \geq 0$ s.t. $D + nH$, $D' + mH$ are hyperplane sections of S.

$\Rightarrow$ Suffices to prove the result for hyperplane sections of S. (in 2 different embeddings)

By generic smoothness (Chap III Cor 10.7), since S nonsingular,

$\exists$ open set $V \subseteq T$ over which g is étale

Can move D & D' s.t. they meet transversely & intersection $\subseteq V$

$\Rightarrow$ g^*D & g^*D' meet transversely

$$\Rightarrow \quad \begin{array}{ccc} g^*D \cap g^*D' & = & g^{-1}(D \cap D') \\ \| & & \| \\ g^*D.g^*D' & & d\,(D.D') \end{array}$$

$\square$

Theorem (Serre duality)

S surface (nonsingular projective $/\mathbb{C}$)
L line bundle on S

ω_S : line bundle of differential 2-forms on S

then $H^2(S, \omega_S)$ is a 1-dim'l $\mathbb{C}$-vector space &

For $0 \leq i \leq 2$, the cup-product pairing

$$H^i(S, L) \otimes H^{2-i}(S, \omega_S \otimes L^{-1}) \longrightarrow H^2(S, \omega_S) \xrightarrow[\sim]{tr} \mathbb{C}$$

is perfect. In particular, $h^i(S, L) = h^{2-i}(S, \omega_S \otimes L^{-1})$

$\chi(L) = \chi(\omega_S \otimes L^{-1})$

Thm (Riemann-Roch formula)

For $\forall\, L \in \operatorname{Pic} S$,

$$\chi(L) = \chi(\mathcal{O}_S) + \frac{1}{2}(L^2 - L.\omega_S)$$

pf. Compute $L^{-1}.(L \otimes \omega_S^{-1})$

by def. of the intersection product

$$L^{-1}.(L \otimes \omega_S^{-1}) = \chi(\mathcal{O}_S) - \chi(L) - \chi(\omega_S \otimes L^{-1}) + \chi(\omega_S) \underset{\text{Serre}}{=} 2\big(\chi(\mathcal{O}_S) - \chi(L)\big)$$

$\square$

Noether's formula

$$\chi(\mathcal{O}_S) = \frac{1}{12}\left(K_S^2 + e(S)\right)$$

where $e(S)$ is the topological Euler-Poincaré characteristic

$$\overset{\shortparallel}{\sum_{i=0}^{4}(-1)^i b_i(S)} \quad \text{with} \quad b_i = \dim_{\mathbb{R}} H^i(S,\mathbb{R})$$

i-th Betti number

genus formula

> C irreducible curve on a surface S
>
> The arithmetic genus of C, defined by $g(C) = h^1(C, \mathcal{O}_C)$
>
> given by
>
> $$g(C) = 1 + \frac{1}{2}(C^2 + C.K_S)$$

pf Consider the structure exact seq.

$$0 \to \mathcal{O}_S(-C) \to \mathcal{O}_S \to \mathcal{O}_C \to 0$$

$$\Rightarrow \underset{1-g(C)}{\underset{\shortparallel}{\chi(\mathcal{O}_C)}} = \chi(\mathcal{O}_S) - \chi(\mathcal{O}_S(-C))$$

$\shortparallel$ R.R.

$$-\frac{1}{2}(C^2 + C.K_S)$$

$\square$

Adjunction formula $\mathcal{O}_S(K_S + C)|_C = \omega_C$ taking degree.

Arithmetic genus of embedded curves.

Def

- C irreducible curve, $\nu : \tilde{C} \to C$ normalization of C then genus $g(\tilde{C})$ of $\tilde{C}$ called the geometric genus of C. $p_g(C) := g(\tilde{C})$
- If $C \subset S$ (nonsingular surface), then the arithmetic genus of C

$$p_a(C) := (1 - \chi(\mathcal{O}_C)) \underset{\uparrow}{=} 1 + \chi(\omega_C)$$

Serre duality for embedded curve

Properties

(1) If C irreducible smooth, then $p_a(C) = p_g(C)$

If $C = \bigcup_{i=1}^{n} C_i$ with

- each C_i smooth irreducible curve,
- all the intersections are transverse
- no three curves meet at a common point.

then $p_a(C) = \sum_{i=1}^{n} p_a(C_i) + \sum_{i<j} (C_i \cdot C_j) - (n-1)$

For example $C = C_1 \cup C_2$, Consider the exact seq.

$$0 \to \mathcal{O}_C \to \mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2} \to \mathcal{O}_{C_1 \cap C_2} \to 0$$

skyscraper sheaf supported on $C_1 \cap C_2$

$$p_a(C) = 1 - \chi(\mathcal{O}_C) = 1 - (\chi(\mathcal{O}_{C_1} \oplus \mathcal{O}_{C_2}) - \chi(\mathcal{O}_{C_1 \cap C_2}))$$
$$= p_a(C_1) + p_a(C_2) + C_1 C_2 - 1$$

Induction on $n =$ # irred. comp. of C

If $C \subset S$ reduced embedded curve, $\nu : \tilde{C} \to C$ normalization

Consider the normalization sequence

$$0 \to \mathcal{O}_C \to \nu_* \mathcal{O}_{\tilde{C}} \to \underset{\uparrow}{\delta} \to 0$$

concentrated on singular points of C

$$\Rightarrow \chi(\mathcal{O}_C) = \chi(\nu_* \mathcal{O}_{\tilde{C}}) - \chi(\delta)$$
$$= \chi(\mathcal{O}_{\tilde{C}}) - \chi(\delta)$$
$$= 1 - \underset{\uparrow}{g(C)} - \sum_{x \in C \text{ singular}} h^0(\delta_x)$$

geometric genus

$$\Rightarrow p_a(C) = g(C) + \sum_{\substack{x \in C \\ \text{singular}}} h^0(\delta_x)$$

In particular, $p_a(C) > p_g(C)$ if C singular

$p_a(C) = 0 \Rightarrow C$ rational & smooth i.e. $C \cong \mathbb{P}^1$

(2) For every embedded curve $C \subset S$, we have (genus formula)

$$p_a(C) = 1 + \frac{1}{2} \deg(K_S \otimes \mathcal{O}_S(C)|_C)$$
$$= 1 + \frac{1}{2}(K_S . C + C^2)$$

In fact, applying Riemann-Roch (for embedded curves)

$$\underset{\substack{\text{dualizing sheaf} \\ \text{(line bundle)} \\ \text{defined by } K_S \otimes \mathcal{O}_S(C)}}{\deg(\omega_C)} = \chi(\omega_C) - \chi(\mathcal{O}_C) \underset{\substack{\text{Serre duality} \\ \text{(for embedded curves)}}}{=\!=} 2\chi(\omega_C)$$

$$\Rightarrow p_a(C) = 1 + \frac{1}{2}\deg\left(K_S \otimes \mathcal{O}_S(C)|_C\right)$$
$$= 1 + \frac{1}{2}(K_S C + C^2).$$

If $C = A+B$, then

$$p_a(A+B) = p_a(A) + p_a(B) + AB - 1$$

So we can define the arithmetic genus for divisors $C=A+B$ as above.

(3) If C reduced & connected, then $p_a(C) \geq 0$

& $p_a(C)=0$ implies that C is a tree of smooth rat'l curves.

here $C = \sum R_i$ is a tree if

- $R_i R_j \leq 1$ for $i \neq j$
- no cycle $R_{i_1}, \cdots, R_{i_n} \subset C$ $(n \geq 3)$ with $R_{i_j} \cdot R_{i_{j+1}} \neq 0$ for $j = 1, \cdots, n-1$ & $R_{i_1} \cdot R_{i_n} \neq 0$
- 3 distinct curves never have a common point

Pf. Since $h^0(\mathcal{O}_C) = 1$ for $\forall$ reduced connected curve C

$$\Rightarrow p_a(C) = 1 - \chi(\mathcal{O}_C) = 1 - h^0(\mathcal{O}_C) + h^1(\mathcal{O}_C) = h^1(\mathcal{O}_C) \geq 0.$$

If $C = \bigcup_{i=1}^{n} C_i$, then $p_a(C) = \sum_{i=1}^{n} p_a(C_i) + \underbrace{\sum_{i<j}(C_i C_j) - (n-1)}_{\geq 0}$

$$p_a(C) = 0 \Rightarrow \underset{(C_i \text{ smooth rational})}{p_a(C_i) = 0} \ \& \ \underset{(C \text{ is a tree of curves})}{\sum_{i<j}(C_i C_j) = n-1}$$

C is a tree of smooth rational curves.

1-Connected divisors

Def | We say an effective divisor C on S is <u>Connected</u> if $\mathrm{Supp}(C)$ is connected.

Rmk For reduced curve C,

$$\text{Connectedness} \Rightarrow h^0(\mathcal{O}_C) = 1.$$

Ramanujam's lemma

| curve $C \subset$ nonsingular surface S

L : line bundle on C with $\deg(L|_{C_i}) = 0$ for $\forall$ irred. comp. $C_i \subset C$

If $s \in H^0(L)$ & $C = C_1 + C_2$ with $C_1 \leq C$ maximal divisor satisfying $s|_{C_1} \equiv 0$, then $C_1 C_2 \leq 0$.

Pf. by assumption $s \in H^0(L \cdot \mathcal{I}_{C_1}) = H^0(L \otimes \mathcal{O}_{C_2}(-C_1))$

then $s : \mathcal{O}_{C_2} \longrightarrow \mathcal{O}_{C_2}(-C_1) \otimes L$ injective

$$0 \longrightarrow \mathcal{O}_{C_2} \xrightarrow{s} \mathcal{O}_{C_2}(-C_1) \otimes L \longrightarrow \underset{\substack{\uparrow \\ \text{has finite support.}}}{\overset{\text{cokernel}}{Q}} \longrightarrow 0$$

Applying Riemann-Roch theorem on C_2, one has

$$\deg(\mathcal{O}_{C_2}(-C_1) \otimes L) = \chi(\mathcal{O}_{C_2}(-C_1) \otimes L) - \chi(\mathcal{O}_{C_2}) = \chi(Q) = h^0(Q) \geq 0$$

$$\deg(\mathcal{O}_{C_2}(-C_1) \otimes L) = \mathcal{O}_S(C_2) . (\mathcal{O}_{C_2}(-C_1) \otimes L) = LC_2 - C_1 C_2 = -C_1 C_2$$

$$\Longrightarrow C_1 C_2 \leq 0.$$

Def | an effective divisor C on a surface S is called m-connected if $C_1 C_2 \geq m$ for each effective decomposition $C = C_1 + C_2$.

Rmks

- 1-Connected also called numerically connected
- 1-Connected $\Rightarrow$ Connected
- Connected $\not\Rightarrow$ 1-Connected, Say $E \subset S$ irreducible curve with $E^2 < 0$ then $2E$ Connected but not 1-Connected

Lemma

C 1-connected curve $\subset S$ (nonsingular surf)

L line bundle on C s.t. $\deg(L|_{C_i}) = 0$ for $\forall$ irred. comp. $C_i \subset C$

then $h^0(L) \leq 1$.

$h^0(L) = 1 \Leftrightarrow L = \mathcal{O}_C$.

Pf. For $s \in H^0(L)$

define $C = C_1 + C_2$ with $C_1 \leq C$ maximal divisor s.t. $s|_{C_1} \equiv 0$.

If $s \neq 0$, then $C_2 \neq 0$.

1-Connectedness of C $\Rightarrow C_1 = 0$.

Consider the exact sequence

$$0 \to \mathcal{O}_{C_2} \xrightarrow{s} \mathcal{O}_{C_2} \otimes L \to \underset{\text{Cokernel has finite support}}{Q} \to 0$$

by Riemann-Roch,

$$h^0(Q) = -C_1C_2 = 0 \Rightarrow Q = 0.$$

$\Rightarrow \mathcal{O}_C \xrightarrow{s} L$ is an isomorphism.

Cor If C 1-connected effective divisor

then $h^0(\mathcal{O}_C) = 1$.

Lemma D 1-connected effective divisor on S

$C \not\subset \mathrm{Supp}(D)$ is an irreducible curve on S with $CD = 1$

then $h^0(\mathcal{O}_D(C)) = 1$, unless the component $R \subset D$ intersecting C is a smooth rational curve

Pf. Write $D = R + E$

then $C \cap \mathrm{Supp}(E) = \phi$ ($CD = 1 \Rightarrow R$ is the only one component of D intersecting C)

Consider the decomposition sequences

$$0 \to \mathcal{O}_E(C-R) \to \mathcal{O}_D(C) \to \mathcal{O}_R(C) \to 0$$
$$\|$$
$$0 \to \mathcal{O}_E(-R) \to \mathcal{O}_D \to \mathcal{O}_R \to 0$$

D 1-connected $\Rightarrow h^0(\mathcal{O}_D) = 1 \Rightarrow H^0(\mathcal{O}_D) \xrightarrow{\text{rest.}} H^0(\mathcal{O}_R)$ injective

$\Downarrow$

$H^0(\mathcal{O}_E(C-R)) = H^0(\mathcal{O}_E(-R)) = 0$

If $h^0(\mathcal{O}_D(C)) > 1$ then $H^0(\mathcal{O}_D(C)) \hookrightarrow H^0(\mathcal{O}_R(C))$ having 2 independent sections

$\mathcal{O}_R(C)$ deg 1 line bundle, which is very ample on R

$\Rightarrow R \xrightarrow{|\mathcal{O}_R(C)|} \mathbb{P}^1$ is an isom.