Cubic surfaces

Set-up: $S \subset \mathbb{P}^3_{[x:y:z:t]}$ smooth cubic surface given by a cubic form $f = f(x,y,z,t)$

Goal: Consider the lines ℓ of $\mathbb{P}^3$ lying on S

1.1. Main tricks

Interested in the lines & triangles of S

here a triangle is a set of 3 distinct coplanar lines $\ell_1, \ell_2, \ell_3 \subset S$, s.t. $\ell_1 + \ell_2 + \ell_3 = S \cap H$ is a hyperplane section.

Classically, we know the following basis facts:

(0) S contains a line ℓ.

a standard argument by dimension counting

$P = \mathbb{P}^N = |\mathcal{O}_{\mathbb{P}^3}(3)|$ projective space of cubics of $\mathbb{P}^3$

↑ $N = h^0(\mathcal{O}_{\mathbb{P}^3}(3)) - 1 = \binom{3+3}{3} - 1 = 19$

$Gr = Gr(2,4)$ Grassmannian of lines in $\mathbb{P}^3$ (set of lines of $\mathbb{P}^3$)

Consider the incidence subvariety.

two projections

$$\begin{tikzcd} & Z = \{(\ell, S) \in Gr \times P \mid \ell \subset S\} \arrow[dl, "p"'] \arrow[dr, "q"] & \\ Gr & & P \end{tikzcd}$$

Choose a homogeneous coordinate $[x:y:z:t]$ for $\mathbb{P}^3$ such that

a cubic surface $S \subset \mathbb{P}^3$ contains a line ℓ (given by $z=t=0$)

$$\Updownarrow$$

the 4 coefficients of x^3, x^2y, xy^2, y^3 in its defining equation ($f(x,y,z,t)=0$) vanish

i.e. Cubics forms vanishing on a given line ℓ form a $\mathbb{P}^4$

$\Rightarrow$ the fibres of $p: Z \to Gr$ have $\dim = \dim P - 4$

&

$\dim Z = \dim P$

If the fact not true, then $\mathrm{Codim}_P\, p(Z) \geq 1$

& the fibre $q^{-1}(S)$ is either empty or positive-dim'l for $\forall S$ in P

Thus $q : Z \longrightarrow P$ is surjective if $\exists$ a cubic S containing a non-empty finite set of lines.

Fact | $\forall$ Cubic del Pezzo surface contains a finite number of lines

$\square$

(1) any two intersecting lines determine a triangle.

(Consequence of nonsingularity of S)

Observation 1 | through $\forall$ point $P \in S$,

$\exists \leqslant 3$ lines of S

if $\exists$ 2 or 3 lines, they must be coplanar.

If $p \in \ell \subset S$, then $\ell = T_p \ell \subset T_p S$

$\Downarrow$

all lines of S through P are contained in the plane $T_p S$. More generally, we have

Observation 2 | $\forall$ plane $\Pi \subset \mathbb{P}^3$ intersects S in one of the following

(i) irreducible cubic (ii) a conic + a line

(iii) 3 distinct lines.

suffices to show that a multiple line is impossible!

if Π $(t=0)$ & $\ell : (z=0)$ $\subset \Pi$, then

ℓ a multiple line of $S \cap \Pi \Leftrightarrow f$ is of the form

$$f = z^2 \underbrace{A(x,y,z,t)}_{\text{linear form}} + t \cdot \underbrace{B(x,y,z,t)}_{\text{quadratic form}}$$

then S ($\overset{\text{defined by}}{f=0}$) singular at a point where $z=t=B=0$

which is a nonempty set.

$\uparrow$ zero locus of B on the line ℓ (i.e. $B(x,y,0,0)=0$)

by assumption, S is nonsingular!

(2) If ℓ_1, ℓ_2, ℓ_3 a triangle of S & M a fourth line of S,

then M meets exactly one of ℓ_1, ℓ_2, ℓ_3.

(by above observations)

(3) For $\forall$ line ℓ on S, there exist exactly 10 disjoint lines of S meeting ℓ, falling into 5 coplanar pairs (ℓ_i, ℓ_i') & the pairs are disjoint $(\ell_i \cup \ell_i') \cap (\ell_j \cup \ell_j') = \phi$ whenever $i \neq j$

(4) In particular, $\exists$ two disjoint lines L, M on S.

1.2 all the lines of S

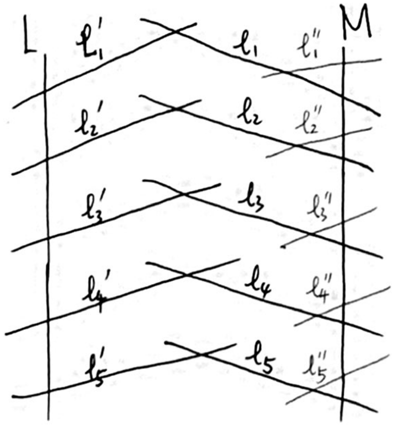
Fix two disjoint lines $l, M \subset S$.

$l \rightsquigarrow 5$ triangles (l, l_i, l_i')

M meets exactly one of l_i, l_i', by renumbering, let these be $l_1, l_2, \cdots, l_5$.

$M \sim 5$ triangles (M, l_i, l_i'')

Claim: $\exists$ 10 further lines $l_{klm} \subset S$ which meet l_k, l_l, l_m & not l_i, l_j for $\{i, j, k, l, m\} = \{1, 2, 3, 4, 5\}$



numbers of lines on S

$15 + 2 + 10 = 27$

need to check that $L, M, l_i, l_i', l_i'', l_{ijk}$ exhaust all lines of S

Proof of claim. If $i \neq j$, then $L_i' \cap M = \phi$ & $L_i' \cap L_j = \phi$

$\Rightarrow L_i' \cap$ (3rd line L_j'' of triangle (M, l_j, l_j'')) $= \phi$

i.e. L_i' meets L_j'' (whenever $i \neq j$)

Choose some L_i', say L_1' & Consider the 5 triangles involving l_1'

- (l, l_1, l_1')
- 4 other triangles determined by intersecting pairs $(l_1', l_j'')_{j \in \{2,3,4,5\}}$

For example, consider the triangle (l_1', l_2'', N)

$\Rightarrow N$ not intersects any line pair of other 4 triangles

In particular, N not intersect l_3'', l_4'', l_5''
$N \cap l_1 = \phi$

(M, l_2, l_2'') triangle $\quad N$ fourth line $\Rightarrow N \cap l_2 = \phi$

$(M, l_3, l_3'') \quad N \Longrightarrow N \cap l_3 \neq \phi$

Similarly $N \cap l_4 \neq \phi$, $N \cap l_5 \neq \phi$ denote $N = l_{345}$ $\square$

1.3 the lattice A(S)

define A(X) to be the free abelian group on the 27 lines of S modulo the relation

$$\ell + \ell' + \ell'' = M + M' + M'' \text{ whenever } \begin{matrix}(\ell,\ell',\ell'') \\ \& \\ (M,M',M'')\end{matrix} \text{ are triangles}$$

<u>Prop</u> $A(X) \cong \mathbb{Z}^7$ with a basis $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_5', \ell_5''$.

<u>Pf</u>. • the 5 triangles containing ℓ are $\ell + \ell_i' + \ell_i$ $(1 \le i \le 5)$

then $\ell + \ell_i + \ell_i' = \ell + \ell_5 + \ell_5' \in A(S)$ $\forall i$

$\Rightarrow \ell_i' = \ell_5 + \ell_5' - \ell_i \in A(S)$ ①

- Similarly, triangles involving $M \Rightarrow \ell_i'' = \ell_5 + \ell_5'' - \ell_i \in A(S)$ ②
- by the (proof of) above claim, $(\ell_i', \ell_j'', \ell_{klm})$ a triangle $\{i,j,k,l,m\} = \{1,2,3,4,5\}$

then $\ell + \ell_i + \ell_i' = \ell_i' + \ell_j' + \ell_{klm}$

$\Rightarrow \ell_{klm} = \ell + \ell_i - \ell_j'' \in A(S)$ ③

- Easy to see $\ell_1 + \ell_{123} + \ell_{145}$ triangle

by ③
$$\left.\begin{aligned} \ell_{123} &= \ell + \ell_4 - \ell_5'' \\ \ell_{145} &= \ell + \ell_2 - \ell_3'' \quad \text{in } A(S) \\ \ell + \ell_1' + \ell_1'' &= \ell_1 + \ell_{123} + \ell_{145} \end{aligned}\right\} \Rightarrow$$

$$\forall\ \ell = -3\ell_1 - \ell_2 - \ell_3 - \ell_4 + 3\cancel{8}\,\ell_5 + \ell_5' + 4\ell_5''$$

Suffices to show that $\ell_1, \ell_2, \ell_3, \ell_4, \ell_5, \ell_5', \ell_5''$ are linearly independent in A(S). (proved later via scalar product on A(S))

1.4 Scalar product on $A(S)$

$\exists$ a scalar product $A(S) \times A(S) \longrightarrow \mathbb{Z}$ s.t.

① for two distinct lines ℓ, ℓ',

$$\ell . \ell' = \begin{cases} 1 & \text{if } \ell \text{ meets } \ell' \\ 0 & \text{otherwise} \end{cases}$$

② for $\forall$ line ℓ,

$$\ell^2 = -1$$

③ for $\forall$ line ℓ, $\forall$ triangle (M, M', M'')

$$\ell\,(M + M' + M'') = 1$$

For ③, L a line, M, M', M'' triangle

- if L distinct from (M, M', M'') then L meets exactly one of M, M', M''

  by ①, $L\,(M + M' + M'') = 1$

- if $L = M$, then

  by ① & ②, $L\,(M + M' + M'') = -1 + 2 = 1$

$\Rightarrow$ scalar product is well-defined on $A(S)$.

Prop The scalar product of 7 elements
$e_0 = L_5 + L_5' + L_5''$
$e_1 = L_1$ $e_2 = L_8$ $e_3 = L_3$
$e_4 = L_5$, $e_5 = L_5'$ $e_6 = L_5''$

is given by $e_0^2 = +1$, $\underset{(1 \leq i \leq 6)}{e_i^2 = -1}$ & $\underset{(i \neq j)}{e_i e_j = 0}$

In particular, $e_0, e_1, \cdots, e_6$ is a basis of $A(S)$.

Pf $\begin{cases} e_1 = \ell_1,\ e_2 = \ell_2,\ e_3 = \ell_3,\ e_4 = \ell_4\ \ e_5 = \ell_5'\ \ e_6 = \ell_5'' \ \text{ 6 disjoint lines} \\ e_1, e_2, e_3, e_4 \text{ are disjoint from } \ell_5 \end{cases}$

$e_0 = \ell_5 + \ell_5' + \ell_5''$

by definition, $e_0^2 = 1$, $\underset{1 \leq i \leq 6}{e_i^2 = -1}$, $\underset{(i \neq j)}{e_i e_j = 0}$

suffices to check identities involving e_0

$$e_0 e_5 = (\ell_5 + \ell_5' + \ell_5'')\,\ell_5' = 1 - 1 + 0 = 0$$

$$e_0 e_6 = (\ell_5 + \ell_5' + \ell_5'')\,\ell_5'' = 0$$

$$e_0^2 = e_0\,(\ell_5 + e_5 + e_6) = e_0 \ell_5 = -1 + 1 + 1 = 1$$

1.5 Symmetric treatment of lines

Notation

h : class of a triangle in $A(S)$

Claim :

$h^2 = 3 \in A(S)$

$hx \equiv x^2 \bmod 2$ for $\forall x \in A(S)$

Pf If ℓ, ℓ', ℓ'' any triangle, then $h = \ell + \ell' + \ell'' \in A(S)$

$$h^2 = h(\ell + \ell' + \ell'') = 1 + 1 + 1 = 3 \in A(S)$$

$$(x^2 \bmod 2) : \begin{array}{ccc} A(S) & \longrightarrow & \mathbb{F}_2 \\ x & \longmapsto & (x^2 \bmod 2) \end{array}$$ is a linear function,

since $(x+y)^2 = x^2 + 2xy + y^2 \equiv x^2 + y^2 \pmod 2$

Suffices to check $hx \equiv x^2 \bmod 2$ for $\forall$ generator $x \in A(S)$.

Note that $A(S)$ generated by lines ℓ of S,

$$h\ell = 1, \ \ell^2 = -1 \Rightarrow h\ell \equiv \ell^2 \bmod 2$$

$\square$

Summary

given a nonsingular cubic surface $S \subset \mathbb{P}^3$, $\exists$ a lattice $A(S) \cong \mathbb{Z}$ with a scalar product $A(S) \times A(S) \to \mathbb{Z}$, which can be diagonalized to $\mathrm{diag}(1, -1, -1, \dots, -1)$

& $\exists$ an element $h \in A(S)$ s.t. $h^2 = 3$, $hx \equiv x^2 \bmod 2$ $\in A(S)$

Ex $e_0, e_1, \dots, e_6$ a basis of $A(S)$

express h, classes of 27 lines in terms of this basis

1.6 divisor class group Pic S

Set-up : X normal var.

a prime divisor on X is an irreducible codim 1 subvar $Y \subseteq X$

a (Weil) divisor D on X is a formal $\mathbb{Z}$-linear combination of prime divisors Y_i on X. say $D = \sum n_i Y_i$

$\mathrm{Div}(X)$: free abelian group generated by prime divisors of X

Call D is effective, if all $n_i \geq 0$. (write $D \geq 0$)

Recall that if $Y \subset X$ prime divisor, then $\mathcal{O}_{Y,X}$ is a 1-dim'l local ring at Y, that is, $\overset{\text{rat'l fct}}{f \in k(X)}$ s.t. f regular at some points of Y

Suppose that X nonsingular, $\mathcal{O}_{Y,X}$ DVR

$\rightsquigarrow$ a discrete valuation $v_Y : k(X)^* \longrightarrow \mathbb{Z}$ s.t.

$$\mathcal{O}_{Y,X} = \{ f \in k(X)^* \mid v_Y(f) \geq 0 \} \cup \{0\}$$

for $f \in k(X)^*$, we say that

$v_Y(f) > 0 \iff f$ has a zero along Y of order $v_Y(f)$

$v_Y(f) < 0 \iff f$ has a pole along Y of order $|v_Y(f)|$

$v_Y(f) = 0 \iff f$ & f^{-1} regular along Y.

Fact. for $0 \neq f \in k(X)$,

$$\mathrm{div}(f) := \sum_{\substack{Y \subset X \\ \text{prime}}} v_Y(f)\, Y \text{ is a divisor on } X$$

a principal divisor is a divisor of form $\mathrm{div}(f)$ for some $f \in k(X)^*$

two divisors D, D' are linearly equivalent if $D - D' = \mathrm{div}(f)$ for some $0 \neq f \in k(x)$ nonzero rat'l fct
(write $D \underset{\text{lin}}{\sim} D'$)

If X nonsingular, its divisor class group $\mathrm{Pic}\, X$ is the group $\mathrm{Div}(X) / \underset{\text{lin}}{\sim}$

Suppose that $D \geq 0$, $D' \geq 0$, D & D' no common components s.t. $D - D' = \mathrm{div}(f)$

$\Rightarrow f$ has zeros on D & poles on D'

Can view f as a rat'l map $f : X \dashrightarrow \mathbb{P}^1$ s.t. $f^{-1}(0) = D$, $f^{-1}(\infty) = D'$

$\Rightarrow$ $D_t = f^{-1}(t)$ is a divisor on X moving with $t \in \mathbb{P}^1$
(a linear pencil of divisors on X)

<u>Example</u>:

$X \subset \mathbb{P}^3$ nonsingular cubic surface

$L + L' + L''$ triangle cut out on X by the plane $A = 0$
(resp. $M + M' + M''$) (resp. $B = 0$)

where A, B two linear forms on $\mathbb{P}^3$
(homog. poly. of deg 1)

then $A/B \in k(X)$ &

$$\mathrm{div}(A/B) = L + L' + L'' - M' - M - M''$$

$\Rightarrow$ any two triangles are linearly equivalent.

$\Rightarrow$ a well-defined map $\partial : A(X) \longrightarrow \mathrm{Pic}\, X$.

(a priori, ∂ bijective)

1.7 Intersection numbers

S : nonsingular projective suface $/\mathbb{C}$ $(k=\bar{k})$

D, D_1, D_2 divisors on S

Can define the intersection numbers of divisors $D_1 . D_2$ s.t.

① $D_1 \cdot D_2$ bilinear in each factor & $D_1D_2 = D_2D_1$
symmetric

② $D_1 D_2$ only depends on D_1, D_2 up to linear equivalence.

③ if $D_1, D_2 \geq 0$ & no common componets

then
$$D_1 D_2 = \sum_{P \in D_1 \cap D_2} (D_1 D_2)_P$$

here
$$(D_1 D_2)_P := \dim_{\mathbb{C}} \frac{\mathcal{O}_{S,P}}{(I_{D_1}+I_{D_2}) \cdot \mathcal{O}_{S,P}} = \dim_{\mathbb{C}} \frac{\mathcal{O}_{S,P}}{(f_1, f_2)}$$

where D_1, D_2 locally (around p) defined by f_1, f_2, resp.

④ if C irreducible curve, then

$$C.D = \deg_C \mathcal{O}_C(D).$$

⇝ intersection numbers of divisor give rise to a bilinear form

$$\mathrm{Pic}\, S \times \mathrm{Pic}\, S \longrightarrow \mathbb{Z}$$

Called the intersection pairing.

recall that the scalar product on $A(S)$ is an intersectio pairing on $\mathrm{Pic}\, S$ via $\partial : A(S) \longrightarrow \mathrm{Pic}\, S$

<u>NB</u>. for irreducible curves C, C'

$$CC' < 0 \Leftrightarrow C = C'.$$

Self-intersection number $C^2 = \deg_C \mathcal{N}_{C/S}$

1.8 Conic bundles & cubic surfaces

$S \subset \mathbb{P}^3$ nonsingular cubic surface

$L \subset S$ a line

define the projection away from L

$$\varphi = \varphi_L : S \dashrightarrow \mathbb{P}^1$$

as follows:

take $M = \mathbb{P}^1 \subset \mathbb{P}^3$, disjoint to L

if $p \in S \setminus L$ then $\langle p, L \rangle$ spans a plane $\Pi = \mathbb{P}^2 \subset \mathbb{P}^3$

Say $\Pi \cap M = t \in \mathbb{P}^1$

define

$$\varphi_L : S \setminus L \longrightarrow \mathbb{P}^1$$
$$p \longmapsto t = \Pi \cap M$$

lemma | $\varphi_L : S \setminus L \longrightarrow \mathbb{P}^1$ can extend to a morphism $\varphi : S \longrightarrow \mathbb{P}^1$

here are two proofs.

(1st proof) let $L = (x = y = 0)$ s.t. S defined by $Ax + By = 0$ where A, B quadratic forms in x, y, z, t.

Which have no common zeros on L (by nonsingularity of S)

So $x : y = -B : A$ everywhere defined

$\rightsquigarrow \varphi : S \longrightarrow \mathbb{P}^1$ a morphism

$$p \longmapsto [-B(p) : A(p)]$$

(2nd pf) Let $H = L + F$ a hyperplane section of S through L.

moving H to a linearly equiv. section, we have

$$HL = 1, \quad HF = 2 = LF$$

$$\Rightarrow H^2 = 3 \ \& \ F^2 = F(H-L) = 0$$

Since the $F \geq 0$ & distinct as H varies $\Rightarrow$ they disjoint

$\varphi : S \longrightarrow \mathbb{P}^1$ well-defined

$$F_t \longmapsto \overset{\rotatebox{-90}{$\in$}}{t}$$

$\varphi^{-1}(t) = F_t$ plane conic curves residual to L.

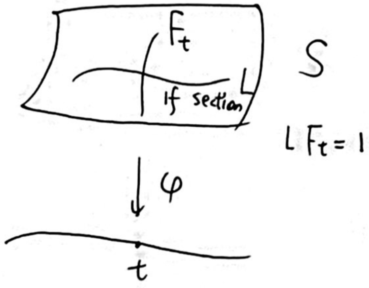
Rmks

- If $F = L_1 + L_2$, $\xRightarrow{F^2=0}$ $0 = F^2 = (L_1+L_2)^2 = \underset{-1}{\underline{\underline{L_1^2}}} + 2L_1L_2 + \underset{-1}{\underline{\underline{L_2^2}}}$

$$\Downarrow$$

$$L_1L_2 = 1$$

- L not a section of the bundle, since $LF = 2$



1.9 Other birat'l modes for S.

given two ~~distinct~~ disjoint lines L, M lying on S.

Can construct

$$\varphi = \varphi_L \times \varphi_M : S \longrightarrow \mathbb{P}^1 \times \mathbb{P}^1$$

recall φ_L Conic bundle obtained as the linear projection away from L.

φ_L : 5 line pairs meeting L $\longmapsto$ 5 distinct points of $\mathbb{P}^1$

φ_L & φ_M : 5 lines $L_1, \cdots, L_5$ meeting both L & M $\longmapsto$ 5 distinct pts of $\mathbb{P}^1 \times \mathbb{P}^1$

Fact S is the blowup of $\mathbb{P}^1 \times \mathbb{P}^1$ over these 5 distinct points $Q_1, \cdots, Q_5 \in \mathbb{P}^1 \times \mathbb{P}^1$.

1.10 Construction of blowup

Suppose first that $S = \mathbb{A}^2$ with coordinates x, y

$P = (0,0)$

define $S_1 \subset \mathbb{A}^2 \times \mathbb{P}^1$ to be the closed graph of the rational map $\mathbb{A}^2 \dashrightarrow \mathbb{P}^1$

$(x,y) \longmapsto [x:y]$

$\overline{\{x, y \,;\, \underset{u}{x} : \underset{v}{y}\}}$

If $(u:v)$ are homogeneous coordinates of $\mathbb{P}^1$

then S_1 defined by $x/u = \frac{y}{v}$ (i.e. $xv = yu$)

$$\begin{tikzcd}
S_1 \arrow[r, hook] \arrow[dr, "\sigma"'] & \mathbb{A}^2 \times \mathbb{P}^1 \arrow[d] \\
& \mathbb{A}^2
\end{tikzcd}$$

So if $(x,y) \neq (0,0)$, then $(u:v)$ well-defined. $\Rightarrow$

$$\begin{cases} \sigma^{-1}(P) = \mathbb{P}^1 \subset S_1 \leftarrow \text{defined by } xv = yu \\ S_1 \setminus \sigma^{-1}(P) \cong S \setminus P. \end{cases}$$

Since $(u;v)$ homogeneous coordinates of $\mathbb{P}^1$.

if $u \neq 0$, may assume $u = 1$. s.t. the open set

$(u \neq 0) \subset S_1$ is the surface $\subseteq \mathbb{A}^3$ with cordinates (x, y, v) defined by $y = xv$.

More generally, if $P \in S$ is any nonsingular point

let x, y be local coordinates at P, shrinking S to a small enough open set, may assume S is affine. &

$m_P = (x, y)$

↑ (maximal ideal of functions vanishing at P in the cordinate ring of S)

let $\sigma : S_1 \to S$ blowing up a nonsingular point $P \in S$

$$\begin{tikzcd}[row sep=small]
S_1 \arrow[r] & S \\
\cup & \rotatebox{90}{$\in$} \\
C \arrow[r, mapsto] & P
\end{tikzcd}$$

curve

then $C \cong \mathbb{P}^1$ & $C^2 = -1$.

Such a curve C called a (-1)-curve.

(Castelnuovo's contractibility criterion)

X nonsingular surface, $C \subset X$ curve

then $\exists$ a morph. $f : X \longrightarrow Y$ to a nonsingular surface & $\begin{cases} \text{isom. outside } C \\ \text{contracting } C \text{ to a pt} \end{cases}$

C a (-1)-curve $\Updownarrow$

1.11 the cubic surfaces obtained as $\mathbb{P}^2$ blown up 6 pts

THEOREM

$\Sigma = \{P_1, \cdots, P_6\} \subset \mathbb{P}^2$ is a set of 6 pts with the following "general position" properties:

(1) no 2 points coincide

(2) no 3 colinear

(3) not all 6 on a conic.

S : blow up of $\mathbb{P}^2$ at the 6 points Σ.

then the vector space of cubic forms on $\mathbb{P}^2$ vanishing at Σ is 4-dim'l & if $F_1, \cdots, F_4$ is a basis, then the rat'l map

$$\begin{aligned} \mathbb{P}^2 &\dashrightarrow \mathbb{P}^3 \\ P &\longmapsto [F_1(P) : \cdots : F_4(P)] \end{aligned}$$

induces an isom of S with a nonsingular cubic surface $S \cong S_3 \subseteq \mathbb{P}^3$.

This construction is a 2-sided inverse to $\psi : S \to \mathbb{P}^2$.